1.12.09

Maß- und Wahrscheinlichkeitsth. UE

VIII,

1) $X_1, \dots, X_n$ i.i.d. $\quad X_{(n)} := \max\{X_1, \dots, X_n\}, \; X_{(1)} := \min\{X_1, \dots, X_n\}$

a) $F_{X_{(n)}}(x) = P(X_{(n)} \le x) = P(X_1 \le x, \dots, X_n \le x)$

$= \prod_{i=1}^{n} P(X_i \le x) = (F(x))^n$

$\Rightarrow f_{X_{(n)}}(x) = n\, F(x)^{n-1} \cdot f(x)$

$F_{X_{(1)}}(x) = P(X_{(1)} \le x) = 1 - P(X_{(1)} > x)$

$= 1 - P(X_1 > x, \dots, X_n > x)$

$= 1 - \prod_{i=1}^{n} P(X_i > x) = 1 - (1 - F(x))^n$

$\Rightarrow f_{X_{(1)}}(x) = n\,(1 - F(x))^{n-1} \cdot f(x)$

b) $F_{X_{(1)}, X_{(n)}}(x,y) = P(X_{(1)} \le x, X_{(n)} \le y)$

$A \cap B = B \setminus (A^c \cap B)$

$= P(X_{(n)} \le y) - P(X_{(1)} > x, X_{(n)} \le y)$

$= F_{X_{(n)}}(y) - \prod_{i=1}^{n} P(x < X_i \le y)$

$= (F(y))^n - (F(y) - F(x))^n \qquad x \le y$

$F_{X_{(1)}, X_{(n)}}(x,y) = (F(y))^n \qquad x > y$

c) $X_{(1)}, X_{(n)}$ unabh. $\Leftrightarrow F_{X_{(1)}, X_{(n)}} = F_{X_{(1)}} \cdot F_{X_{(n)}}$

$F_{X_{(1)}}(x) \cdot F_{X_{(n)}}(y) = F(y)^n (1 - (1 - F(x))^n)$

$= F(y)^n - (F(y) - F(y)F(x))^n \neq F_{X_{(1)}, X_{(n)}}(x,y)$

d) $X_1, X_2 \sim U_{0,1}$

$\Rightarrow f_{X_1}(x) = f_{X_2}(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{sonst} \end{cases}, \quad F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$

$F_{X_{(1)}, X_{(2)}}(x,y) = \begin{cases} F(y)^2 - (F(y) - F(x))^2 = 2F(y)F(x) - F(x)^2 & x \le y \\ F(y)^2 & x > y \end{cases}$

$\Rightarrow F_{X_{(1)}, X_{(2)}}(x,y) = \begin{cases} 0 & x \wedge y < 0 \\ 2x - x^2 & 0 \le x \le 1 \wedge y > 1 \\ 2yx - x^2 & 0 \le x \le y \le 1 \\ y^2 & 0 \le y \le x \wedge y \le 1 \\ 1 & x \wedge y > 1 \end{cases}$

2) a) $PX^{-1}(x_1,\dots,x_k) = P(X_1,\dots,X_k)^{-1}(x_1,\dots,x_k) = P(X_1=x_1,\dots,X_k=x_k)$

$$= \prod_{i=1}^{k} P\Big(X_i=x_i \,\Big|\, \bigcap_{j=1}^{i-1} X_j=x_j\Big)$$

$$= \prod_{i=1}^{k} \binom{n-\sum_{j=1}^{i-1} x_j}{x_i} p_i^{x_i}$$

$$\prod_{i=1}^{k} \binom{n-\sum_{j=1}^{i-1} x_j}{x_i} = \prod_{i=1}^{k} \frac{(n-\sum_{j=1}^{i-1} x_j)!}{x_i!\,(n-\sum_{j=1}^{i} x_j)!}$$

$$= n! \prod_{i=1}^{k-1} \frac{(n-\sum_{j=1}^{i} x_j)!}{x_i!\,(n-\sum_{j=1}^{i} x_j)!} \cdot \frac{1}{x_k!\,(n-\sum_{j=1}^{k} x_j)!}$$

$$= \frac{n!}{\prod_{i=1}^{k} x_i!}$$

$$\Rightarrow P(X_1,\dots,X_k)^{-1}(x_1,\dots,x_k) = n! \prod_{i=1}^{k} \frac{p_i^{x_i}}{x_i!}$$

b) $PX^{-1}_{i_1,\dots,i_m}(x_{i_1},\dots,x_{i_k}) = P(X_{i_1},\dots,X_{i_m})^{-1}(x_{i_1},\dots,x_{i_m})$

$$= n! \prod_{j=1}^{m} \frac{p_{i_j}^{x_{i_j}}}{x_{i_j}!} \cdot \frac{\left(1-\sum_{j=1}^{m} p_{i_j}\right)^{n-\sum_{j=1}^{m} x_{i_j}}}{\left(n-\sum_{j=1}^{m} x_{i_j}\right)!}$$

3) $U \sim U_{0,1}$, F Verteilungsfkt. $\Rightarrow R := F^{-1}\circ U \sim F$

$$f(x) = 2ax^{-ax^2} \Rightarrow F(x) = \int f(x)\,dx = 2a\int x e^{-ax^2}\,dx$$

$$= \left| \begin{matrix} x^2 = y \\ 2x\,dx = dy \end{matrix} \right| = a\int e^{-ay}\,dy = -e^{-ax^2} + c$$

$F(x) \in [0,1]\ \forall x \in \mathbb{R} \Rightarrow c = 1.$

$a, x > 0$: $\quad e^{-ax^2} = 1 - F(x) \Leftrightarrow -ax^2 = \ln(1-F(x))$

$$\Leftrightarrow x = a^{-\frac{1}{2}}\sqrt{-\ln(1-F(x))}$$

$\Rightarrow F^{-1}(y) = a^{-\frac{1}{2}}\sqrt{-\ln(1-y)} \quad y \in [0,1)$

$R = F^{-1}\circ U = a^{-\frac{1}{2}}\sqrt{-\ln(1-U)} = a^{-\frac{1}{2}}\sqrt{-\ln U} \sim$ Rayleigh

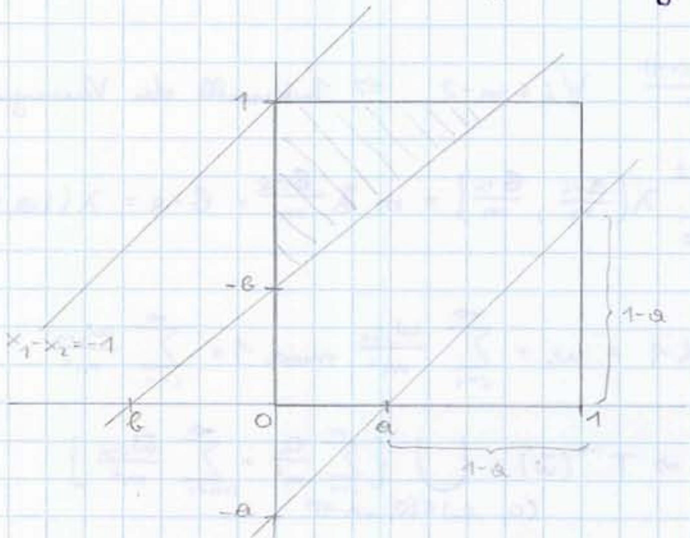
4) $T: ([0,1]^2, \mathcal{L}_2) \to (\mathbb{R}, \mathcal{L})$

$$(x_1, x_2) \mapsto x_1 - x_2$$

$\Rightarrow T([0,1]^2) = [-1,1]$

$$F_T(a) = \lambda_2 T^{-1}((-\infty, a]) = \lambda_2 (T^{-1}([-1,a]))$$

$$= \lambda_2 (\{(x_1,x_2) \in [0,1]^2 \mid x_1 - x_2 \leq y\})$$



$$\Rightarrow F_T(a) = \begin{cases} 0 & a < -1 \\ \frac{(1+a)^2}{2} & -1 \leq a < 0 \\ 1 - \frac{(1-a)^2}{2} & 0 \leq a \leq 1 \\ 1 & a > 1 \end{cases}$$

$$\Rightarrow f_T(a) = \begin{cases} 1+a & -1 \leq a < 0 \\ 1-a & 0 \leq a \leq 1 \\ 0 & \text{sonst} \end{cases} \Leftrightarrow \begin{cases} 1 - |a| & -1 \leq a \leq 1 \\ 0 & \text{sonst} \end{cases}$$

$$\lambda_2 T^{-1}([-a,a]) = F_T(a) - F_T(-a) = 1 - \frac{(1-a)^2}{2} - \frac{(1-a)^2}{2} = 2a - a^2$$

5) $(\Omega, \mathfrak{A}) = ([0,1), \mathcal{L} \cap [0,1], P)$, $P(\{\omega\}) = 0 \quad \forall \omega \in [0,1)$

$P(\{\omega\}) = F(\omega) - F(\omega)^- = 0 \Rightarrow$ P stetig, nur Intervalle haben WS.

•) $T_1(\omega) = \alpha\omega \quad \alpha \in (0,1) \Rightarrow T_1^{-1}(\varpi) = \frac{\varpi}{\alpha}$

$$\Rightarrow P(T_1^{-1}([a,b))) = P([\tfrac{a}{\alpha}, \tfrac{b}{\alpha})) \underset{\text{z.B. für } P=\lambda}{=} \frac{1}{\alpha} P([a,b)) \neq P([a,b)) \qquad a < b \leq \alpha$$

•) $T_2(\omega) = \omega^2 \Rightarrow T_2^{-1}(\varpi) = \sqrt{\varpi}$

$$\Rightarrow P(T_2^{-1}([a,b))) = P([\sqrt{a}, \sqrt{b})) \neq P([a,b)) \qquad \forall [a,b) \subseteq [0,1).$$

also: T_1, T_2 nicht maßtreu.

6) $T(\omega) := m\omega \bmod 1 := m\omega - \lfloor m\omega \rfloor \qquad m \in \mathbb{N},\ m \geq 2$

ZZ: T auf $([0,1), \mathcal{L} \cap [0,1), \lambda)$ maßtreu und mischend.

$$T(\omega) = \begin{cases} m\omega & \omega < \frac{1}{m} \\ m\omega - 1 & \omega \in [\frac{1}{m}, \frac{2}{m}) \\ \vdots \end{cases} \Rightarrow T(\omega) = m\omega - i \quad \omega \in [\tfrac{i}{m}, \tfrac{i+1}{m}) \quad i < m$$

$$\Downarrow$$

$$T^{-1}(\bar{\omega}) = \frac{\bar{\omega}+i}{m} \in [\tfrac{i}{m}, \tfrac{i+1}{m})$$

$$\Rightarrow T^{-1}((a,b]) = \bigcup_{i=0}^{m-1} \left(\frac{a+i}{m}, \frac{b+i}{m}\right] \quad \forall 0 \leq a \leq b < 1$$

$$\frac{b+i}{m} < \frac{1+i}{m} < \frac{a+(i+1)}{m} \quad \forall i \leq m-2 \Rightarrow \text{Intervalle der Vereinigung disjunkt.}$$

$$\Rightarrow \lambda T^{-1}((a,b]) = \sum_{i=0}^{m-1} \lambda\left(\frac{a+i}{m}, \frac{b+i}{m}\right] = m \frac{b-a}{m} = b-a = \lambda((a,b]) \Rightarrow T \text{ maßtreu!}$$

Sei $\omega = \sum_{i=1}^{\infty} \frac{\omega_i}{m^i}$, $\omega_i \in \{0, .., m-1\}$:

$$T(\omega) = \sum_{i=1}^{\infty} \frac{m\omega_i}{m^i} \bmod 1 = \omega_1 + \sum_{i=1}^{\infty} \frac{\omega_{i+1}}{m^i} \bmod 1 = \sum_{i=1}^{\infty} \frac{\omega_{i+1}}{m^i}$$

$$\Rightarrow T^n(\omega) = \sum_{i=1}^{\infty} \frac{\omega_{i+n}}{m^i} \Rightarrow T^{-n}(\bar{\omega}) = \bigcup_{(c_1,\dots,c_n) \in \{0,\dots,m-1\}^n} \left\{ \sum_{i=1}^{n} \frac{c_i}{m^i} + \sum_{i=n+1}^{\infty} \frac{\bar{\omega}_{i-n}}{m^i} \right\}$$

T ist mischend $\Leftrightarrow \lim_{n\to\infty} \lambda(A \cap T^{-n}(B)) = \lambda(A) \cdot \lambda(B) \quad \forall A, B \in \mathcal{T}$ mit $a_\sigma(\mathcal{T}) = \mathcal{L} \cap [0,1)$

Es genügt also, m-adische Intervalle zu betrachten:

$$A = \left[\sum_{i=1}^{k} \frac{a_i}{m^i}, \sum_{i=1}^{k} \frac{a_i}{m^i} + \frac{1}{m^k}\right),\ B = \left[\sum_{i=1}^{l} \frac{b_i}{m^i}, \sum_{i=1}^{l} \frac{b_i}{m^i} + \frac{1}{m^l}\right) \qquad a_i, b_i \in \{0, \dots, m-1\}$$

Sei $n \geq k$ (auf endl. viele Glieder kommt es nicht an), dann gilt:

$$A \cap T^{-n}(B) = \left[\sum_{i=1}^{k} \frac{a_i}{m^i}, \sum_{i=1}^{k} \frac{a_i}{m^i} + \frac{1}{m^k}\right) \cap \bigcup_{(c_1,\dots,c_n) \in \{0,\dots,m-1\}^n} \left[\sum_{i=1}^{n} \frac{c_i}{m^i} + \sum_{i=n+1}^{n+l} \frac{b_{i-n}}{m^i}, \sum_{i=1}^{n} \frac{c_i}{m^i} + \sum_{i=n+1}^{n+l} \frac{b_{i-n}}{m^i} + \frac{1}{m^{n+l}}\right)$$

$$= \bigcup_{(c_{k+1},\dots,c_n) \in \{0,\dots,m-1\}^{n-k}} \left[\sum_{i=1}^{n+l} \frac{c_i}{m^i}, \sum_{i=1}^{n+l} \frac{c_i}{m^i} + \frac{1}{m^{n+l}}\right) \quad \text{mit } c_1 = a_1, \dots, c_k = a_k; c_{n+1} = b_1, \dots, c_{n+l} = b_l$$

$$\Rightarrow \lambda(A \cap T^{-n}(B)) = \sum_{\substack{(c_{k+1},\dots,c_n) \\ \in \{0,\dots,m-1\}^{n-k}}} \frac{1}{m^{n+l}} = \frac{m^{n-k}}{m^{n+l}} = \frac{1}{m^{k+l}} = \frac{1}{m^k} \cdot \frac{1}{m^l} = \lambda(A) \cdot \lambda(B) \Rightarrow T \text{ mischend!}$$

7) $X_1, X_2, \dots$ i.i.d. $X_i \sim B_p \Rightarrow P(X_i = 1) = p,\ P(X_i = 0) = 1-p$

$$L_k := \max\{i \mid \prod_{j=0}^{i} X_{k+j} = 1\}$$

$$P(L_k = i) = P\left(\prod_{j=0}^{i} X_{k+j} = 1 \wedge X_{k+i+1} = 0\right) = P(X_k = 1, \dots, X_{k+i} = 1, X_{k+i+1} = 0)$$

$$= \prod_{j=0}^{i} P(X_j = 1) \cdot P(X_{k+i+j} = 0) = p^{i+1}(1-p)$$

$$\mathbb{E} L_R = \sum_{i=0}^{\infty} i \, P(L_R = i) = \sum_{i=0}^{\infty} i \, p^{i+1} (1-p)$$

$$= p^2 (1-p) \sum_{i=1}^{\infty} i \, p^{i-1}$$

$$= p^2 (1-p) \left(\sum_{i=0} p^i \right)'$$

$$= p^2 (1-p) \left(\frac{1}{1-p} \right)' = \frac{p^2}{1-p}$$