10.6.09

Lin. Algebra UE

XII) 12.9: 1, 2, 5, 7
12.10: 2, 3, 7
13.1: 1

12.9.1) a) $(\mathbb{R}^{3\times 1}, \iota)$

$$a_1 = \begin{pmatrix}1\\1\\2\end{pmatrix} \quad a_2 = \begin{pmatrix}3\\-1\\1\end{pmatrix}$$

$$\det G(a_1, a_2) = \begin{vmatrix} a_1 \cdot a_1 & a_1 \cdot a_2 \\ a_2 \cdot a_1 & a_2 \cdot a_2 \end{vmatrix} = \begin{vmatrix} 6 & 4 \\ 4 & 11 \end{vmatrix} = 50$$

$\Rightarrow$ Flächeninhalt: $\pm 5 \cdot \sqrt{2}$

b) $(\mathbb{R}^{4\times 1}, \iota)$

$$a_1 = \begin{pmatrix}2\\1\\0\\-1\end{pmatrix} \quad a_2 = \begin{pmatrix}1\\0\\1\\0\end{pmatrix} \quad a_3 = \begin{pmatrix}-2\\1\\1\\0\end{pmatrix}$$

$$\det G(a_1, a_2, a_3) = \begin{vmatrix} 6 & 2 & -3 \\ 2 & 2 & -1 \\ -3 & -1 & 6 \end{vmatrix} = 72 + \cancel{6} - \cancel{6} - 18 - 6 - 24 = 36$$

$\Rightarrow$ Rauminhalt: ± 6

12.9.2) $(\mathbb{R}^{4\times 1}, \iota)$ $\quad \iota(E, E) = \operatorname{diag}(1, 1, 2, 3)$

$$a = \begin{pmatrix}1\\0\\2\\1\end{pmatrix} \quad b = \begin{pmatrix}0\\2\\1\\0\end{pmatrix} \quad c = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$G(a, b, c) = \begin{pmatrix} a \cdot a & b \cdot a & a \cdot c \\ b \cdot a & b \cdot b & b \cdot c \\ c \cdot a & c \cdot b & c \cdot c \end{pmatrix} = \begin{pmatrix} 12 & 4 & 3 \\ 4 & 6 & 0 \\ 3 & 0 & 3 \end{pmatrix}$$

$$\det G(a, b, c) = 12 \cdot 18 - 3 \cdot 18 - 4 \cdot 12 = 114$$

12.9.5) $(\mathbb{V}, v)$ n-dim. eukl.

$B = (b_1, \dots, b_r)$ Basis von U (UR von $\mathbb{V}$)

$pr: \mathbb{V} \to U$ orth. Proj. auf U

ZZ: VS $\Rightarrow \det G(a, b_1, b_2, \dots, b_r) = \det G(b_1, b_2, \dots, b_r) \cdot \|a - pr(a)\|^2 \quad \forall a \in \mathbb{V}$

$$a = \underset{\in U}{u} + \underset{\in U^\perp}{u_\perp} = \sum_{i=1}^{r} x_i b_i + u_\perp \qquad \forall a \in V$$

$$pr(a) = u \Rightarrow u_\perp = a - u = a - pr(a)$$

$$\Rightarrow \det G(a, b_1, \dots, b_r) = \det G\left(\sum_{i=1}^{r} x_i b_i + u_\perp, b_1, \dots, b_r\right)$$

$$= \sum_{i=1}^{r} x_i \cdot \underbrace{\det G(b_i, b_1, \dots, b_r)}_{=0\ \forall i} + \det G(u_\perp, b_1, \dots, b_r)$$

$$= \begin{vmatrix} u_\perp \cdot u_\perp & \overbrace{\boxed{u_\perp \cdot b_1 \ \dots \ u_\perp \cdot b_r}}^{=0} \\ \underbrace{\boxed{\begin{matrix} b_1 \cdot u_\perp \\ \vdots \\ b_r \cdot u_\perp \end{matrix}}}_{=0} & \begin{matrix} b_1 \cdot b_1 & \dots & b_1 \cdot b_r \\ \vdots & \ddots & \vdots \\ b_r \cdot b_1 & \dots & b_r \cdot b_r \end{matrix} \end{vmatrix}$$

Entw. n. 1. Spr.

$$= \|u_\perp\|^2 \cdot \det G(b_1, \dots, b_r) = \|a - pr(a)\|^2 \cdot \det G(b_1, \dots, b_r)$$

12.10.2) a)

$$A = (1, 1, 1, 1) \qquad \Rightarrow f_A: \mathbb{R}^{4 \times 1} \to \mathbb{R}$$

<u>Singulärwertzerlegung:</u>

$$\underset{\in \mathbb{R}^{1\times 4}}{A} = \underset{\in O_1}{Q} \cdot \underset{\in \mathbb{R}^{1\times 4}}{\begin{pmatrix} W & 0 \\ \cancel{0} & \cancel{0} \end{pmatrix}} \cdot \underset{\in O_4}{P}$$

$$O_1 = \{(1)\} \Rightarrow Q = E_1 = 1$$

$$P^T = \langle \underset{\text{Basis v. } \mathbb{R}^{4\times 1}}{E^*}, B \rangle = (\underset{\in (\mathrm{Ker} f_A)^\perp}{b_1}, \underbrace{b_2, b_3, b_4}_{\in \mathrm{Ker} f_A})$$

$$\mathrm{Ker} f_A = A^\circ = \left[\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right] \qquad (\mathrm{Ker} f_A)^\perp = \left[\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right]$$

$$\Rightarrow P^T = \frac{1}{2} \begin{pmatrix} 1 & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 1 & \sqrt{2} & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 \\ 1 & 0 & 0 & \sqrt{2} \end{pmatrix}$$

$$A = Q \cdot (w\ 0\ 0\ 0) \cdot P \Leftrightarrow P^T \cdot \begin{pmatrix} w \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot 1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow w = 2$$

$$\Leftrightarrow A^T = P^T \cdot (\text{——} \cdot \text{——})^T \cdot Q^T \Rightarrow$$

Pseudoinverse:

also $W = (2,0,0,0) \Rightarrow W^{+} = (\frac{1}{2},0,0,0)^T$

$\Rightarrow A^{+} = P^T \cdot W^{+} \cdot Q^T$

$$= \frac{1}{4}\begin{pmatrix} 1 & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 1 & \sqrt{2} & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 \\ 1 & 0 & 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

b) $B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \Rightarrow f_B : \mathbb{R}^{3\times 1} \to \mathbb{R}^{2\times 1}$

Singulärwertezerlegung:

$$\underset{\in \mathbb{R}^{2\times 3}}{B} = \underset{\in O_2}{Q} \cdot \underset{\in \mathbb{R}^{2\times 3}}{\begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}} \cdot \underset{\in O_3}{P}$$

$\ker f_B = B^0 = \left[\underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{b_3}\right]$

$(\ker f_B)^{\perp} = \left[\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{b_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{b_2}\right] \Rightarrow f(\mathbb{R}^{3\times 1}) = f((b_1, b_2)) = \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] =: Q$

$P^T = (b_1, b_2, b_3) \Rightarrow P = P^T = E_3$

$\Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \overset{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}{=} \begin{pmatrix} w_1 & 0 & 0 \\ 0 & w_2 & 0 \end{pmatrix} \qquad \Rightarrow w_1 = w_2 = 1$

also $W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow W^{+} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

Pseudoinverse:

$\Rightarrow B^{+} = P^T \cdot W^{+} \cdot Q^T$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$

c) $C = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \Rightarrow f_C : \mathbb{R}^{2\times 1} \to \mathbb{R}^{2\times 1}$

Singulärwertezerlegung:

$$C = \underset{\in O_2}{Q} \cdot \operatorname{diag}(w_1, w_2) \cdot \underset{\in O_2}{P}$$

$\ker f_C = C^0 = \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] \Rightarrow b_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$(\ker f_C)^{\perp} = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] \Rightarrow b_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow f_C(b_1) = \begin{pmatrix} \frac{2}{\sqrt{2}} \\ 0 \end{pmatrix} \Rightarrow Q := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E_2$

$P^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = P$

$$\Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} w_1 = \sqrt{2} \\ w_2 = 0 \end{matrix}$$

Pseudoinverse:

$$C^{\dagger} = P^T \cdot W^{\dagger} \cdot Q^T$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{pmatrix}$$

12.10.3) a) $A \in K^{m \times n}$, Spalten bilden ONS von $\mathbb{K}^{m \times 1}$

n Spalten, l.u. $\Rightarrow$ rg $A = n$

$$\overset{(12.36)}{\Rightarrow} A^{\dagger} = (\underbrace{\bar{A}^T \cdot A}_{= \begin{pmatrix} a_1 \cdot a_1 & a_2 \cdot a_1 \cdots \\ a_2 \cdot a_1 & a_2 \cdot a_2 \cdots \\ \vdots & \end{pmatrix} = E_n})^{-1} \cdot \bar{A}^T = \bar{A}^T$$

b) $N_{m,n} :=$ $m \times n$ Nullmatrix über $\mathbb{K}$

Singulärwertzerlegung:

Singulärwerte ... Eigenwerte von $N_{mn}^T \cdot N_{mn} = N_{nm} \cdot N_{mn} = N_{nn}$

EW v. N_{nn}: 0 (Vielfachheit n)

$$\Rightarrow N_{mn} = \underset{=: Q \in O_m(\mathbb{K}, \omega)}{E_m} \cdot N_{mn} \cdot \underset{=: P \in O_n(\mathbb{K}, \omega)}{E_n} \qquad \Rightarrow W = \text{Nullmatrix}$$

Pseudoinverse:

$$N_{mn}^{\dagger} = \bar{P}^T \cdot N_{nm} \cdot \bar{Q}^T = E_n \cdot N_{nm} \cdot E_m = N_{nm}$$

12.10.7)
$$\begin{aligned} a x_1 + a x_2 &= 2a \\ x_1 + 2 x_2 &= 3 \\ x_1 + x_2 &= 3 \end{aligned} \qquad A = \begin{pmatrix} a & a \\ 1 & 2 \\ 1 & 1 \end{pmatrix} \qquad (s_i) = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix}$$

rg $A = 2$ $\forall a \in \mathbb{R}$ $\Rightarrow$ löse Normalgleichungssystem $A^T A \cdot (x_j) = A^T (s_i)$

$$A^T A = \begin{pmatrix} a & 1 & 1 \\ a & 2 & 1 \end{pmatrix} \begin{pmatrix} a & a \\ 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a^2+2 & a^2+3 \\ a^2+3 & a^2+5 \end{pmatrix} \qquad A^T (s_i) = \begin{pmatrix} 2a^2+6 \\ 2a^2+9 \end{pmatrix}$$

x_1	x_2	1	
a^2+2	a^2+3	$2a^2+6$	$\cdot -1$
a^2+3	a^2+5	$2a^2+9$	
a^2+2	a^2+3	$2a^2+6$	$\cdot -(a^2+2)$
1	2	3	
0	$-a^2-1$	$-a^2$	$\cdot -\frac{1}{a^2+1}$
1	2	3	
0	1	$\frac{a^2}{a^2+1}$	$\cdot -2$
1	2	3	
1	0	$3 - 2\frac{a^2}{a^2+1}$	

$$\Rightarrow \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = \begin{pmatrix} 3 - 2\frac{a^2}{a^2+1} \\ \frac{a^2}{a^2+1} \end{pmatrix}$$

13.1.1) $\mathcal{A} = \mathbb{C}^{2\times 1}$ ι... kan. unitäres SKP

Σ Sphäre, Gleichung: $(x-m)\cdot(x-m) = 1$

$m = \begin{pmatrix} 0 \\ m_2 \end{pmatrix}$ Gerade g: $x_2 = 0$

ges.: $\Sigma \cap g$

$$(x-m)\cdot(x-m) = \overline{(x-m)}^T \cdot (x-m)$$
$$= \overline{(x_1-m_1)}(x_1-m_1) + \overline{(x_2-m_2)}(x_2-m_2)$$
$$= \overline{x_1}\cdot x_1 + \overline{(x_2-m_2)}(x_2-m_2) = 1$$

$\Rightarrow \Sigma \cap g$: $\overline{x_1}\cdot x_1 + \overline{m_2}\cdot m_2 = |x_1|^2 + |m_2|^2 = 1$

also $\Sigma \cap g = \{x_1 \in \mathbb{C} \mid |x_1| = \sqrt{1-|m_2|^2}\}$

„Ellipse" in Gaußscher Zahlenebene