27.5.09

Lin. Algebra UE

X) 12.3: 1, 4
12.4: 2, 3a, 7
12.5: 2, 4, 8

12.3.1) $(\mathbb{R}^{3\times1}, \iota)$ $\quad \iota(E,E) = \operatorname{diag}(1,1,-1)$

$$\underbrace{\langle E^*, f(E)\rangle}_{f \in L(\mathbb{R}^{3\times1}, \mathbb{R}^{3\times1})} = \begin{pmatrix} 0 & -2 & -4 \\ 2 & 0 & -1 \\ -4 & -1 & 0 \end{pmatrix}$$

ges.: $\langle E^*, \hat{f}(E)\rangle$, ZZ: f normal.

$$\hat{f} = d^{-1}_{\mathbb{R}^{3\times1}} \circ f^T \circ d_{\mathbb{R}^{3\times1}} \Leftrightarrow \langle E^*, \hat{f}(E)\rangle = (\iota(E,E))^{-1} \cdot \langle E^*, f(E)\rangle^T \cdot \iota(E,E)$$

$$\langle E^*, \hat{f}(E)\rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & +2 & -4 \\ -2 & 0 & 1 \\ -4 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 4 \\ -2 & 0 & 1 \\ 4 & 1 & 0 \end{pmatrix}$$

$$\langle E^*, f(E)\rangle\langle E^*, \hat{f}(E)\rangle = \begin{pmatrix} -12 & -4 & -2 \\ -4 & 3 & 8 \\ 2 & -8 & -17 \end{pmatrix} = \langle E^*, \hat{f}(E)\rangle\langle E^*, f(E)\rangle$$

$\Rightarrow f \circ \hat{f} = \hat{f} \circ f$, d.h. f normal.

12.4.3a) $A := \frac{1}{7}\begin{pmatrix} -3 & 2 & 6 \\ -6 & -3 & -2 \\ 2 & -6 & 3 \end{pmatrix}$

Drehwinkel φ:

$$\cos\varphi = \frac{\operatorname{sp} A - 1}{2} = \left(-\frac{3}{7} - 1\right)\cdot\frac{1}{2} = -\frac{5}{7} \Rightarrow \varphi = \arccos\left(-\frac{5}{7}\right) = 0{,}7532\ldots\pi$$
$$\approx 135{,}6^\circ$$

Drehachse $[b_1]$:

$$A + A^T - (\operatorname{sp} A - 1)E_3$$

$$= \frac{1}{7}\left(\begin{pmatrix} -3 & 2 & 6 \\ -6 & -3 & -2 \\ 2 & 6 & 3 \end{pmatrix} + \begin{pmatrix} -3 & -6 & 2 \\ 2 & -3 & 6 \\ 6 & -2 & 3 \end{pmatrix} - \begin{pmatrix} -10 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -10 \end{pmatrix}\right) = \frac{1}{7}\begin{pmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{pmatrix}$$
$$\underbrace{}_{\text{l.a.}}$$

$$\Rightarrow [b_1] = \left[\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right]$$

d.h. $f_A(x) = x \quad \forall x \in [b_1]$

$\sphericalangle(x, f_A(x)) = \varphi \quad \forall x \in [b_1]^\perp \setminus \{0\}$

$\Rightarrow$ A beschreibt Drehung.

12.5.2) a) $pr: \mathbb{V} \to \mathbb{V}$ Projektion.

ZZ: $\exists \hat{pr}: \mathbb{V} \to \mathbb{V}$ zu pr adjungiert $\Rightarrow \hat{pr}$ Projektion.

pr Projektion $\Rightarrow pr \circ pr = pr \Rightarrow \hat{pr} = \widehat{pr \circ pr} = \hat{pr} \circ \hat{pr} \Rightarrow \hat{pr}$ Projektion.

b) ZZ: $\forall pr$ Orthogonalprojektion $\Leftrightarrow pr$ selbstadjungiert

„$\Rightarrow$" wurde bereits in Satz 12.5.3 gezeigt

„$\Leftarrow$" $pr = \hat{pr} \Rightarrow (pr(\mathbb{V}))^\perp = \ker \hat{pr} = \ker pr \Rightarrow pr(\mathbb{V}) \perp \ker pr$, also pr Ortho-Proj.

c) ~~ZZ~~ Sei $\mathbb{V}$ anisotrop, pr Projektion.

ZZ: pr ist Orthogonalprojektion $\Leftrightarrow pr$ normal

„$\Rightarrow$" pr Orthoproj. $\overset{b)}{\Rightarrow} pr = \hat{pr} \Rightarrow pr \circ \hat{pr} = pr \circ pr = \hat{pr} \circ pr$

$\Rightarrow pr$ normal

„$\Leftarrow$" pr normal: $pr(a) \cdot pr(a) = a \cdot (\hat{pr} \circ pr)(a) = a \cdot (pr \circ \hat{pr})(a) = \hat{pr}(a) \cdot \hat{pr}(a)$

$\Rightarrow \|pr(a)\| = \|\hat{pr}(a)\| \Rightarrow \ker pr = \ker \hat{pr}$

$\Rightarrow \ker pr = (pr(\mathbb{V}))^\perp$

$\mathbb{V}$ anisotrop $\Rightarrow \mathbb{V} = \ker pr \oplus pr(\mathbb{V}) \Rightarrow pr$ ist Ortho.-Proj.

12.5.4) $A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \in \mathbb{R}^{3\times3}$ $\qquad A_2 = \begin{pmatrix} 1 & i & -i \\ -i & 2 & 0 \\ i & 0 & 2 \end{pmatrix} \in \mathbb{C}^{3\times3}$

ges.: $P \in O_3$: $P^{-1} A_1 P = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$

$$\chi_{A_1} = \det(A_1 - XE_3) = \begin{vmatrix} 1-X & 0 & 1 \\ 0 & 1-X & 1 \\ 1 & 1 & -X \end{vmatrix} = (-X)(1-X)^2 - (1-X) - (1-X) = (1-X)(X^2 - X - 2) = (1-X)(X+1)(X-2)$$

$\Rightarrow \lambda_1 = 1, \; \lambda_2 = -1, \; \lambda_3 = 2$

EV zu λ_1: LGS:
$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array} \Rightarrow b_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

EV zu λ_2:
$$\begin{array}{ccc|c} 2 & 0 & 1 & \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & \end{array} \Rightarrow b_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

EV zu λ_3:
$$\begin{array}{ccc|c} -1 & 0 & 1 & \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -2 & \end{array} \Rightarrow b_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow$ Basis aus EV ist OGB, Transformation in ONB durch Normierung:

$\|b_1\| = \sqrt{2}, \quad \|b_2\| = \sqrt{6}, \quad \|b_3\| = \sqrt{3}$

$$\Rightarrow P := \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3} & 1 & \sqrt{2} \\ -\sqrt{3} & 1 & \sqrt{2} \\ 0 & -2 & \sqrt{2} \end{pmatrix} \in O_3$$

ges: $Q \in U_3: \; Q^{-1} A_2 Q = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$

$$\chi_{A_2} = \det(A_2 - XE_3) = \begin{vmatrix} 1-X & i & -i \\ -i & 2-X & 0 \\ i & 0 & 2-X \end{vmatrix} = (1-X)(2-X)^2 - (2-X) - (2-X)$$
$$= (2-X)(X^2 - 3X + 2 - 2)$$
$$= X(2-X)(-3+X)$$

$\Rightarrow \lambda_1 = 0, \; \lambda_2 = 2, \; \lambda_3 = 3$

EV zu λ_1: LGS
$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \hline 1 & i & -i & \\ -i & 2 & 0 & 0 \\ i & 0 & 2 & \\ \hline 0 & -i & i & \\ -i & 2 & 0 & 0 \\ 0 & 2 & 2 & \end{array} \Rightarrow c_1 = \begin{pmatrix} 2i \\ -1 \\ 1 \end{pmatrix}$$

EV zu λ_2:
$$\begin{array}{ccc|c} -1 & i & -i & \\ -i & 0 & 0 & 0 \\ i & 0 & 0 & \end{array} \Rightarrow c_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

EV zu λ_3:
$$\begin{array}{ccc|c} -2 & i & -i & \\ -i & -1 & 0 & 0 \\ i & 0 & -1 & \end{array} \Rightarrow c_3 = \begin{pmatrix} i \\ 1 \\ -1 \end{pmatrix}$$

$\|c_1\| = \sqrt{6}, \quad \|c_2\| = \sqrt{2}, \quad \|c_3\| = \sqrt{3}$

$$\Rightarrow Q := \frac{1}{\sqrt{6}} \begin{pmatrix} 2i & 0 & \sqrt{2}i \\ -1 & \sqrt{3} & \sqrt{2} \\ 1 & \sqrt{3} & \sqrt{2} \end{pmatrix} \in U_3$$

12.4.7) $K = GF(4), \; V = GF(4)^{2\times 1}, \; \iota(E, E) = E_2$

a) $\omega = id_{GF(4)^{2\times 1}}$

(i) ges.: alle ONB, d.h. $B = (\underset{(x_1,x_2)^T}{b_1}, \underset{(y_1,y_2)^T}{b_2})$ mit

$b_1 \perp b_2 \iff b_1 \cdot b_2 = 0 \iff x_1 y_1 + x_2 y_2 = 0 \iff x_1 y_1 = x_2 y_2$

$\|b_1\| = \sqrt{x_1^2 + x_2^2} = 1 \iff x_1^2 + x_2^2 = 1$

$\|b_2\| = 1 \iff y_1^2 + y_2^2 = 1$

$$x_1^2 + x_2^2 = 1 \iff \begin{cases} x_1^2 = 1, & x_2^2 = 0 \\ x_1^2 = 0, & x_2^2 = 1 \\ x_1^2 = a, & x_2^2 = b \\ x_1^2 = b, & x_2^2 = a \end{cases} \iff \begin{matrix} x_1 = 1, & x_2 = 0 \\ x_1 = 0, & x_2 = 1 \\ x_1 = b, & x_2 = a \\ x_1 = a, & x_2 = b \end{matrix}$$

$$\Rightarrow \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{=: M_1}, \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{=: M_2}, \underbrace{\begin{pmatrix} a & b \\ b & a \end{pmatrix}}_{=: M_3}, \underbrace{\begin{pmatrix} b & a \\ a & b \end{pmatrix}}_{=: M_4}$$

da alle anderen Basen mit $\|b_1\| = \|b_2\| = 1$ die $\perp$-Relation nicht erfüllen.

(ii) Verknüpfungstafel von $O_2(GF(4))$:

$O_2(GF(4)) = \{M_1, M_2, M_3, M_4\}$; $\quad (k) := \begin{pmatrix} k & k \\ k & k \end{pmatrix} \; \forall k \in GF(4)$

$+$	M_1	M_2	M_3	M_4
M_1	(0)	(1)	(b)	(a)
M_2	(1)	(0)	(a)	(b)
M_3	(b)	(a)	(0)	(1)
M_4	(a)	(b)	(1)	(0)

$\cdot$	M_1	M_2	M_3	M_4
M_1	M_1	M_2	M_3	M_4
M_2	M_2	M_1	M_4	M_3
M_3	M_3	M_4	M_1	M_2
M_4	M_4	M_3	M_2	M_1

b) $\omega : x \longmapsto x^2$

(i) ges: alle ONB, d.h. $\mathbb{B} = (b_1, b_2)$ mit

$$b_1 \perp b_2 \iff x_1^2 y_1 + x_2^2 y_2 = 0 \iff x_1^2 y_1 = x_2^2 y_2$$

$$\|b_1\| = 1 \iff x_1^3 + x_2^3 = 1$$
$$\|b_2\| = 1 \iff y_1^3 + y_2^3 = 1$$

$$x_1^3 = \begin{cases} 0 & x_1 = 0 \\ 1 & x_1 \neq 0 \end{cases} \Rightarrow x_1^3 + x_2^3 = 1 \iff \begin{cases} x_1 = 0, & x_2 \neq 0 \\ x_1 \neq 0, & x_2 = 0 \end{cases}$$

$$x_1 \neq 0 \Rightarrow x_2 = 0 \overset{\perp}{\Rightarrow} y_1 = 0 \Rightarrow y_2 \neq 0 \quad \text{o.B.d.A}$$

$$\Rightarrow O_2(GF(4), \omega) = \left\{ \begin{pmatrix} k_{11} & 0 \\ 0 & k_{22} \end{pmatrix} \middle| \; k_{11}, k_{22} \in GF(4) \setminus \{0\} \right\}$$

$$\cup \left\{ \begin{pmatrix} 0 & k_{12} \\ \dots \end{pmatrix} \middle| \; \dots \right\}$$

(ii) Verknüpfungstafeln von $\mathbb{O}_2(GF(4), \omega)$

$$\left(\begin{array}{c|c|c} + & \begin{pmatrix} k_{11} & 0 \\ 0 & k_{22} \end{pmatrix} & \begin{pmatrix} 0 & k_{12} \\ k_{21} & 0 \end{pmatrix} \\ \hline \begin{pmatrix} k_{11} & 0 \\ 0 & k_{22} \end{pmatrix} & (0) & \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \\ \hline \begin{pmatrix} 0 & k_{12} \\ k_{21} & 0 \end{pmatrix} & \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} & (0) \end{array}\right) \qquad \begin{array}{c|cc} \cdot & \begin{pmatrix} k_{11} & 0 \\ 0 & k_{22} \end{pmatrix} & \begin{pmatrix} 0 & k_{12} \\ k_{21} & 0 \end{pmatrix} \\ \hline \begin{pmatrix} k_{11} & 0 \\ 0 & k_{22} \end{pmatrix} & \begin{pmatrix} k_{11}^2 & 0 \\ 0 & k_{22}^2 \end{pmatrix} & \begin{pmatrix} 0 & k_{11}k_{12} \\ k_{22}k_{21} & 0 \end{pmatrix} \\ \hline \begin{pmatrix} 0 & k_{12} \\ k_{21} & 0 \end{pmatrix} & \begin{pmatrix} 0 & k_{12}k_{22} \\ k_{11}k_{21} & 0 \end{pmatrix} & \begin{pmatrix} k_{12}k_{21} & 0 \\ 0 & k_{12}k_{21} \end{pmatrix} \end{array}$$

12.5.8) $f: \mathbb{R}^{3\times 1} \longmapsto \mathbb{R}^{3\times 1}$

$$\langle E^*, f(E)\rangle = \frac{1}{4}\begin{pmatrix} 1 & 5 & \sqrt{2}+4 \\ -3 & 1 & \sqrt{2}+8 \\ \sqrt{2}-4 & \sqrt{2}-8 & 2 \end{pmatrix}$$

A 9.4.2 c): Jede Matrix $\in K^{n\times n}$ kann eindeutig als Summe einer symm.
und einer schiefsymm. Matrix dargestellt werden.

Satz 12.3.5: $\langle B^*, f(B)\rangle$ symm. $\Leftrightarrow$ f selbstadj.
——"—— schiefsymm. $\Leftrightarrow$ f antiselbstadj.

$$\langle E^*, f_1(E)\rangle \overset{\text{vgl. A 9.4.2 b)}}{=} \frac{1}{2}\left(\langle E^*, f(E)\rangle + \langle E^*, f(E)\rangle^T\right)$$

$$= \frac{1}{8}\begin{pmatrix} 2 & 2 & 2\sqrt{2} \\ 2 & 2 & 2\sqrt{2} \\ 2\sqrt{2} & 2\sqrt{2} & 4 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 2 \end{pmatrix}$$

$$\langle E^*, f_2(E)\rangle = \frac{1}{2}\left(\langle E^*, f(E)\rangle - \langle E^*, f(E)\rangle^T\right)$$

$$= \frac{1}{8}\begin{pmatrix} 0 & 8 & 8 \\ -8 & 0 & 16 \\ -8 & 16 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 2 \\ -1 & -2 & 0 \end{pmatrix}$$