20.5.09

Lin. Algebra UE

IX) 11.5.: 2 $\delta\varepsilon$, 6, 8
12.1.: 2, 3, Satz 12.1.9
12.2.: 1, 3

11.5.2 δ) $\mathbb{C}^{3\times 1}$ unitär. $b_1 = \begin{pmatrix} i \\ \sqrt{2}i \\ i \end{pmatrix}$ $b_2 = \begin{pmatrix} i \\ 0 \\ i \end{pmatrix}$ $b_3 = \begin{pmatrix} 0 \\ -i \\ \sqrt{2}i \end{pmatrix}$

<u>Orthogonalisierungsverfahren</u> von Schmidt:

$$a_1 = b_2$$

$$a_2 = b_1 - \frac{a_1 \cdot b_1}{a_1 \cdot a_1} a_1 = \begin{pmatrix} i \\ \sqrt{2}i \\ i \end{pmatrix} - \frac{2}{2}\begin{pmatrix} i \\ 0 \\ i \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2}i \\ 0 \end{pmatrix}$$

$$a_3 = b_3 - \frac{a_1 \cdot b_3}{a_1 \cdot a_1} a_1 - \frac{a_2 \cdot b_3}{a_2 \cdot a_2} = \begin{pmatrix} 0 \\ -i \\ \sqrt{2}i \end{pmatrix} - \frac{\sqrt{2}}{2}\begin{pmatrix} i \\ 0 \\ i \end{pmatrix} + \frac{\sqrt{2}}{2}\begin{pmatrix} 0 \\ \sqrt{2}i \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 0 \\ i \end{pmatrix}$$

$\Rightarrow$ <u>OGB</u>: $\left\{\begin{pmatrix} i \\ 0 \\ i \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{2}i \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 0 \\ i \end{pmatrix}\right\}$

<u>ONB</u> $= \{\tilde{a}_1, \tilde{a}_2, \tilde{a}_3\}$ mit $\tilde{a}_i = \frac{1}{\|a_i\|} a_i$

$$= \left\{\frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 0 \\ i \end{pmatrix}, \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 0 \\ i \end{pmatrix}\right\}$$

ε) $\mathbb{C}^{3\times 1}$ unitär. $b_1 = \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}$ $b_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $b_3 = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}$

$$a_1 = b_2$$

$$a_2 = b_3 \quad \perp$$

$$a_3 = b_1 - \frac{a_1 \cdot b_1}{a_1 \cdot a_1} a_1 - \frac{a_2 \cdot b_1}{a_2 \cdot a_2} a_2 = \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix} - \frac{i}{1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}$$

$\Rightarrow$ OGB: $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} \cdot \frac{1}{2}\right\}$

$\Rightarrow$ ONB: $\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix}\right\}$

11.5.6, $V = C^0([-1,1])$ $\qquad \iota: (f,g) \mapsto \int_{-1}^{1} f(x)\, g(x)\, dx$

$F = \{ f_i : x \mapsto x^i \mid i \in \{0,1,2,3\} \}$ $\qquad [F] = U_3 \subset C^0([-1,1])$

$= \{ f_0, f_1, f_2, f_3 \} = \{ 1, x, x^2, x^3 \}$

a) $\iota(f,g) = \int_{-1}^{1} f_i(x)\, f_j(x)\, dx = \int_{-1}^{1} x^i\, x^j\, dx = \left. \frac{x^{i+j+1}}{i+j+1} \right|_{-1}^{1}$

$$\Rightarrow \iota(F,F) = \begin{pmatrix} 2 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{2}{5} \\ \frac{2}{3} & 0 & \frac{2}{5} & 0 \\ 0 & \frac{2}{5} & 0 & \frac{2}{7} \end{pmatrix} \qquad f_j = \begin{array}{l} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{array} \text{ j-te Komponente}$$

1. <u>OGB</u>

$$e_0 = f_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$e_1 = f_1 - \frac{\overbrace{e_0 \cdot f_1}^{0}}{e_0 \cdot e_0} e_0 = f_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$e_2 = f_2 - \frac{e_0 \cdot f_2}{e_0 \cdot e_0} e_0 - \frac{\overbrace{e_1 \cdot f_2}^{0}}{e_1 \cdot e_1} e_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{\frac{\cancel{2}^{1}}{3}}{\cancel{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$e_3 = f_3 - \frac{\overbrace{e_0 \cdot f_3}^{0}}{e_0 \cdot e_0} e_0 - \frac{e_1 \cdot f_3}{e_1 \cdot e_1} e_1 - \frac{\overbrace{e_2 \cdot f_3}^{0}}{e_2 \cdot e_2} e_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{\frac{\cancel{2}}{5}}{\frac{2}{3}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{3}{5} \\ 0 \\ 1 \end{pmatrix}$$

2. <u>ONB</u>

$$C := \left\{ \frac{1}{\|e_0\|} e_0,\ \frac{1}{\|e_1\|} e_1,\ \frac{1}{\|e_2\|} e_2,\ \frac{1}{\|e_3\|} e_3 \right\}$$

$\|e_0\| = \sqrt{2} \qquad \|e_3\| = \sqrt{\frac{8}{45}}$

$\|e_1\| = \sqrt{\frac{2}{3}} \qquad \|e_4\| = \sqrt{\frac{8}{175}}$

$$\Rightarrow C = \left\{ \frac{1}{\sqrt{2}},\ \sqrt{\frac{2}{3}}\, x,\ \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3} \right),\ \sqrt{\frac{175}{8}} \left(x^3 - \frac{3}{5} x \right) \right\}$$

b) $p: C_0([-1,1]) = U_2 \oplus U_2^\perp \longrightarrow U_2:$

$$e^x \longmapsto \sum_{i=0}^{2} \frac{c_i \cdot e^x}{\underbrace{c_i \cdot c_i}_{=1}} c_i = \sum_{i=0}^{2} (c_i \cdot e^x) c_i$$

$$c_0 \cdot e^x = \frac{1}{\sqrt{2}} \int_{-1}^{1} e^x = \frac{1}{\sqrt{2}} (e - e^{-1})$$

$$c_1 \cdot e^x = \sqrt{\tfrac{3}{2}} \int_{-1}^{1} x e^x \, dx = \sqrt{\tfrac{3}{2}} \Big(x e^x \Big|_{-1}^{1} - \underbrace{\int_{-1}^{1} e^x dx}_{e - e^{-1}} \Big) = \sqrt{\tfrac{3}{2}} \cdot 2e^{-1}$$

$$c_2 \cdot e^x = \sqrt{\tfrac{45}{8}} \int_{-1}^{1} (x^2 - \tfrac{1}{3}) e^x \, dx = \sqrt{\tfrac{45}{8}} \Big((x^2 - \tfrac{1}{3}) e^x \Big|_{-1}^{1} - 2 \underbrace{\int_{-1}^{1} x e^x dx}_{2e^{-1}} \Big)$$

$$= \sqrt{\tfrac{45}{8}} \left(\tfrac{2}{3}(e - e^{-1}) - 4e^{-1} \right) = \sqrt{\tfrac{45}{8}} \, \tfrac{2}{3} \left(e - 7e^{-1} \right)$$

$$\Rightarrow p(e^x) = \frac{1}{2}(e - e^{-1}) + \frac{3}{\cancel{2}} \cancel{2} e^{-1} x + \frac{\overset{15}{\cancel{45}}}{4\cancel{8}} \frac{\cancel{2}}{\cancel{3}} (e - 7e^{-1})(x^2 - \tfrac{1}{3})$$

$$= \left(\frac{15}{4} e - \frac{105}{4} e^{-1} \right) x^2 + 3e^{-1} x - \frac{3}{4} e + \frac{33}{4} e^{-1}$$

$$\approx 0{,}5367 x^2 + 1{,}1036 x + 0{,}9963$$

11.5.8) $(V, \cdot)$ eukl. o. unitär.

a) <u>Ungleichung von Bessel</u>

$(a_i)_{i \in I}$ ONS von V, $J \subset I$ mit $\dim J < \infty$

$$\Rightarrow \sum_{j \in J} |a_j \cdot x|^2 \leq \|x\|^2 \quad \forall x \in V$$

Bew.: $x = \sum_{i \in I} x_i a_i + x_\perp$

$$\sum_{j \in J} |a_j \cdot x|^2 = \sum_{j \in J} \Big| a_j \cdot \Big(\sum_{i \in I} x_i a_i + x_\perp \Big) \Big|^2$$

$$= \sum_{j \in J} \Big| \sum_{i \in I} a_j \cdot x_i a_i + a_j \cdot x_\perp \Big|^2$$

$$= \sum_{j \in J} \Big| \sum_{i \in I} x_i (a_j \cdot a_i) \Big|^2 = \sum_{j \in J} |x_j (a_j \cdot a_j)|^2 = \sum_{j \in J} |x_j|^2$$

$$\|x\|^2 = \Big\| \sum_{i \in I} x_i a_i + x_\perp \Big\|^2 \geq \Big\| \sum_{i \in I} x_i a_i \Big\|^2 = \sum_{i \in I} |x_i|^2 \underbrace{(a_i \cdot a_i)}_{1}$$

$$\geq \sum_{i \in J} |x_i|^2$$

b) <u>Identität von Parseval</u>

$$(b_i)_{i \in I} \text{ ONB von } V \Rightarrow x \cdot y = \sum_{i \in I} (x \cdot b_i)(b_i \cdot y) \quad \forall x, y \in V$$

$$x = \sum_{j \in I} x_j b_j \qquad y = \sum_{j \in I} y_j b_j$$

$$\sum_{i\in I}(x\cdot b_i)(b_i\cdot y) = \sum_{i\in I}\left(\left(\sum_{j\in I} x_j b_j\right)\cdot b_i\right)\left(b_i\cdot\left(\sum_{j\in I} y_j b_j\right)\right)$$
$$= \sum_{i\in I}\left(\sum_{j\in I}\overline{x_j}\,(b_j\cdot b_i)\right)\left(\sum_{j\in I} y_j\,(b_i\cdot b_j)\right)$$
$$= \sum_{i\in I}\overline{x_i}\,y_i = x\cdot y$$

12.2.1 a) Sei $f\in L(\mathbb{V},\mathbb{W})$ mit $x\cdot x = f(x)\cdot f(x)$ $\forall x\in\mathbb{V}$

(i) ZZ: $\omega = id_K \wedge \operatorname{Char} K \neq 2 \Rightarrow f$ isom.

$$(x+y)\cdot(x+y) = f(x+y)\cdot f(x+y)$$
$$\Leftrightarrow \quad \cancel{x\cdot x}+y\cdot x+x\cdot y+\cancel{y\cdot y} = \cancel{f(x)\cdot f(x)}+f(x)\cdot f(y)+f(y)\cdot f(x)+\cancel{f(y)\cdot f(y)}$$
$$\Leftrightarrow \quad 2\,(x\cdot y) = 2\cdot(f(x)\cdot f(y))$$
$$\Leftrightarrow \quad x\cdot y = f(x)\cdot f(y) \quad \forall x,y\in\mathbb{V}$$

(ii) ZZ: $K=\mathbb{C} \wedge \omega = \overline{} \Rightarrow f$ isom.

$$(x+y)\cdot(x+y) = f(x+y)\cdot f(x+y)$$
$$\Leftrightarrow \quad x\cdot y+\overline{x\cdot y} = f(x)\cdot f(y)+\overline{f(x)\cdot f(y)}$$
$$\Leftrightarrow \quad \cancel{2}\operatorname{Re}(x\cdot y) = \cancel{2}\operatorname{Re}(f(x)\cdot f(y)) \qquad \forall x,y\in\mathbb{V}$$
$$(x+iy)\cdot(x+iy) = f(x+iy)\cdot f(x+iy)$$
$$\Leftrightarrow \quad i\,(x\cdot y) - i\,(y\cdot x) = i\,(f(x)\cdot f(y)) - i\,(f(y)\cdot f(x))$$
$$\Leftrightarrow \quad i\,(x\cdot y-\overline{x\cdot y}) = i\,(f(x)\cdot f(y)-\overline{f(x)\cdot f(y)})$$
$$\Leftrightarrow \quad \cancel{2}\operatorname{Im}(x\cdot y) = \cancel{2\operatorname{Re}(x\cdot y)}\ \cancel{2}\operatorname{Im}(f(x)\cdot f(y)) \qquad \forall x,y\in\mathbb{V}$$

b) ZZ: $f_\rho\in L(\mathbb{V},\mathbb{W})$ mit $\rho\in\operatorname{Aut}(K)$ und $x\cdot y = f(x)\cdot f(y)$ $\forall x,y\in\mathbb{V}$

$\Rightarrow \rho = id_K$

Bew.: $c\in K \Rightarrow (cx)\cdot y = f(cx)\cdot f(y)$

$$\Leftrightarrow \omega(c)(x\cdot y) = \omega(\rho(c))(f(x)\cdot f(y))$$
$$\Rightarrow \omega(c) = \omega(\rho(c)) \Rightarrow c = \rho(c) \Rightarrow \rho = id_K.$$

12.2.3) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

zz: $A \approx B$, aber nicht ortho. ähnlich

A ist Matrix in JNF $\Rightarrow$ wir zeigen, dass $P \in Gl_3(\mathbb{R})$ ($\hat{=}$ Matrix der Haupttraum-Basen)

existiert, sodass $A = P^{-1} B P$, aber $P \notin O_3$

$\chi_B(X) = (X-1)(X-2)^2 \Rightarrow \lambda_1 = 1,\ \lambda_2 = 2$

EV zu ~~E~~W1: $(1,0,0)^T =: c_1$

EV zu EW2:

x_1	x_2	x_3	
-1	0	1	0
0	0	1	0
0	0	0	0

$\Rightarrow c_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow$ Hauptraum zum EW 2:

x_1	x_2	x_3	
-1	0	1	0
0	0	1	1
0	0	0	0

$\Rightarrow c_3 := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow P := \langle C^*, f(C) \rangle = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$P \notin O_3$ da $c_1 \not\perp c_3 \Rightarrow A, B$ nicht orthogonal ähnlich.

12.1.2) $\dim V = n < \infty$, $f \in L(V,V)$

ges.: Zusammenhang zw. $\chi_f(X)$ und $\chi_{\hat{f}}(X)$.

B bel. Basis von V
$\hat{B}$ zu B rezipr. Basis

$\omega(\langle B^*, f(B) \rangle^T) = \omega(\langle \hat{f}(B^*), B \rangle^T) \overset{\omega\text{-symm.}}{=} \langle B^*, \hat{f}(B) \rangle$

$\langle \hat{B}^*, \hat{f}(\hat{B}) \rangle = \underbrace{\langle \hat{B}^*, B \rangle}_{P^{-1}} \langle B^*, \hat{f}(B) \rangle \underbrace{\langle B^*, \hat{B} \rangle}_{P}$

$\Rightarrow \langle \hat{B}^*, \hat{f}(\hat{B}) \rangle \approx \omega(\langle B^*, f(B) \rangle^T)$

$\Rightarrow \chi_{\hat{f}}(X) = \omega(\chi_f(X))$, da $\det A = \det A^T$.