10.2.1

α) $\lambda : \mathbb{R}^{2\times 1} \to \mathbb{R} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto 2x_1^2 + 2x_1x_2 + 6x_1 + x_2^2 + 2x_2 + 7$

$\Rightarrow q_0 : \mathbb{R}^{3\times 1} \to \mathbb{R} : \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \mapsto 2x_1^2 + 2x_1x_2 + 6x_0x_1 + x_2^2 + 2x_0x_2 + 7x_0^2$

$$\begin{array}{ccc} 7 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \xmapsto{s} \begin{array}{ccc} 6 & 2 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{array} \xmapsto{z} \begin{array}{c|cc} 6 & 2 & 0 \\ \hline 2 & 1 & 0 \\ 0 & 0 & 1 \\ \hline \end{array} \xmapsto{s} \begin{array}{ccc} 2 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{array}$$

$$\xmapsto{z} \begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{array} \qquad \begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline \frac{1}{\sqrt{2}} & 0 & 0 \\ -\sqrt{2} & 1 & 0 \\ \frac{1}{\sqrt{2}} & -1 & 1 \end{array} \; E_3$$

Koo.-Darstellung von λ bez. $(\tilde{u}, \tilde{u} + \tilde{B})$ mit $\tilde{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $\tilde{B} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

$\lambda \circ (E^* \circ \tau_{\tilde{u}})^{-1} : \mathbb{R}^{2\times 1} \to \mathbb{R} : \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} \mapsto \tilde{x}_1^2 + \tilde{x}_2^2 + 2$

γ) $\lambda : \mathbb{R}^{2\times 1} \to \mathbb{R} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto 2x_1^2 + 4x_1 + 2x_2^2 + 2$

$\Rightarrow q_0 : \mathbb{R}^{3\times 1} \to \mathbb{R} : \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \mapsto 2x_1^2 + 4x_0x_1 + 2x_2^2 + 2x_0^2$

$$\begin{array}{ccc} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \xmapsto{s} \begin{array}{ccc} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ \hline 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \xmapsto{z} \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ \hline \end{array} \xmapsto{s} \xmapsto{z} \begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ -1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{array}$$

$(\tilde{u}, \tilde{u} + B)$ mit $\tilde{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $B = \frac{1}{\sqrt{2}} \cdot E_2$

$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} \mapsto \tilde{x}_1^2 + \tilde{x}_2^2$

λ) $q : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto 2x_1x_2 - 2x_1x_3 - 2x_1 + x_2^2 + 2x_2 - 6x_3 - 1$

$\Rightarrow q_0 : \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto 2x_1x_2 - 2x_1x_3 - 2x_0x_1 + x_2^2 + 2x_0x_2 - 6x_0x_3 - 1x_0^2$

$\cdot -1$

$$\begin{array}{cccc} -1 & -1 & 1 & -3 \\ -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 \\ -3 & -1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \xmapsto{S} \begin{array}{cccc} -1 & \cancel{0}\,{-2} & 1 & -3 \\ -1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ -3 & -1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \cdot -1 \xmapsto{Z} \begin{array}{cccc} -1 & -2 & 1 & -3 \\ -2 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ -3 & -1 & 0 & 0 \\ \hline \end{array}$$

$\cdot -1$

$$\xmapsto{S} \begin{array}{cccc} -1 & -2 & 1 & -1 \\ -2 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -3 & -1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array} \cdot -1 \xmapsto{Z} \begin{array}{c|ccc} -1 & -2 & 1 & -1 \\ \hline -2 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ \hline \end{array} \cdot +1$$

$\cdot -1$

$$\begin{array}{c|ccc} -2 & -2 & 1 & 0 \\ \hline \cancel{-2} & \cancel{-2} & \cancel{1} & \cancel{-1} \\ -2 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline \end{array}$$

$$\xmapsto{S} \begin{array}{ccccc} -2 & \cancel{0} & -2 & 1 & -1 \\ -2 & \cancel{-2} & -1 & 0 & 0 \\ 1 & \cancel{1} & 0 & 1 & 0 \\ 0 & \cancel{-1} & 0 & 0 & 1 \\ \hline 1 & \cancel{1} & 0 & 0 & 0 \\ -1 & \cancel{1} & 1 & 0 & -1 \\ 1 & \cancel{-1} & -1 & 1 & 1 \\ 1 & \cancel{-1} & 0 & 0 & 1 \end{array} \cdot -1 \xmapsto{Z}$$

$\cdot 2$

$$\xmapsto{S} \begin{array}{cccc} -3 & -2 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{array} \cdot -1 \xmapsto{Z} \begin{array}{cccc} -3 & -2 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline \end{array}$$

$$\begin{array}{cccc} 1 & -2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & \cancel{1} \\ -2 & -1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{array} \Big) \cdot 2 \quad \overset{S}{\longmapsto} \quad \overset{Z}{\longmapsto} \quad \begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & +1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & -1 \\ -2 & -1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{array}$$

$$(\tilde{u}, \tilde{u} + \tilde{B}) \text{ mit } \tilde{u} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad \tilde{B} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{pmatrix} \mapsto x_1^2 + x_2^2 - x_3^2 + 1$$

10.2.3

a) zz: $\lambda: \mathcal{A} = r + X \to K$ qu. Fkt. $\Rightarrow \lambda|_{\mathcal{A}_1}: \overset{\subseteq \mathcal{A}}{\mathcal{A}_1} = r_1 + X_1 \to K$ qu. Fkt.

$$\lambda(x) = \underbrace{q(x-A)}_{X \to K} + 2\langle \underbrace{q^*, x-A}_{X \to K} \rangle + q \quad \forall x \in \mathcal{A}, \text{ speziell } \forall x \in \mathcal{A}_1$$

wähle $\tilde{A} \in \mathcal{A}_1$:

$$\overset{(10.2)}{\Rightarrow} \underset{= \lambda(x)|_{\mathcal{A}_1}}{\cancel{\lambda(x)}} = \underbrace{q(x-\tilde{A})}_{X' \to K} + 2\langle \underbrace{\tilde{q}^*, x-\tilde{A}}_{X' \to K} \rangle + \tilde{q} \quad \forall x \in \mathcal{A}_1$$

b) zz:
$$\left.\begin{array}{l} \alpha: \mathcal{A} \to \mathcal{A}' \text{ affine Abb.} \\ \lambda': \mathcal{A}' \to K \text{ qu. Fkt.} \end{array}\right\} \Rightarrow \lambda' \circ \alpha: \mathcal{A} \to K \text{ qu. Fkt.}$$

$$\left.\begin{array}{l} \alpha(\mathcal{A}) \subseteq \mathcal{A}' \text{ und } \alpha(\mathcal{A}) \text{ affiner UR } (6.3.5) \\ \lambda' \text{ qu. Fkt.} \end{array}\right\} \overset{a)}{\Rightarrow} \lambda'|_{\alpha(A)} = \lambda' \circ \alpha \text{ qu. Fkt.}$$

10.3.2

$k_1: \begin{cases} x_2 = 0 \\ 2x_3 = x_1^2 \end{cases}$ $\qquad k_2: \begin{cases} x_1 = 0 \\ 2a x_3 = x_2^2 \end{cases} \qquad a \in K^\times$

a) ZZ: $\Phi(\lambda) = \{ x \in K^{3\times 1} \mid x = x_1 + x_2 ,\ x_1 \in k_1, x_2 \in k_2 \}$ ist Paraboloid

$$x_1 = \begin{pmatrix} x_1 \\ 0 \\ \frac{x_1^2}{2} \end{pmatrix} \qquad x_2 = \begin{pmatrix} 0 \\ x_2 \\ \frac{x_2^2}{2a} \end{pmatrix}$$

$$\Rightarrow x = x_1 + x_2 = \begin{pmatrix} x_1 \\ x_2 \\ \frac{1}{2}\left(x_1^2 + \frac{x_2^2}{a}\right) \end{pmatrix}$$

$\Rightarrow$ Gleichung von x: $\quad x_3 = \frac{1}{2}\left(x_1^2 + \frac{x_2^2}{a}\right)$

$$\Leftrightarrow 2x_3 = x_1^2 + \frac{1}{a} x_2^2$$

$\Rightarrow \underline{\lambda(x) = x_1^2 + \frac{1}{a} x_2^2 - 2x_3}$ ... Paraboloid

b) $a > 0 \quad \Rightarrow \lambda(x) = x_1^2 \oplus \frac{1}{a} x_2^2 - 2x_3$ ... elliptisch

$a < 0 \quad \Rightarrow \lambda(x) = x_1^2 \ominus \frac{1}{|a|} x_2^2 - 2x_3$ ... hyperbolisch

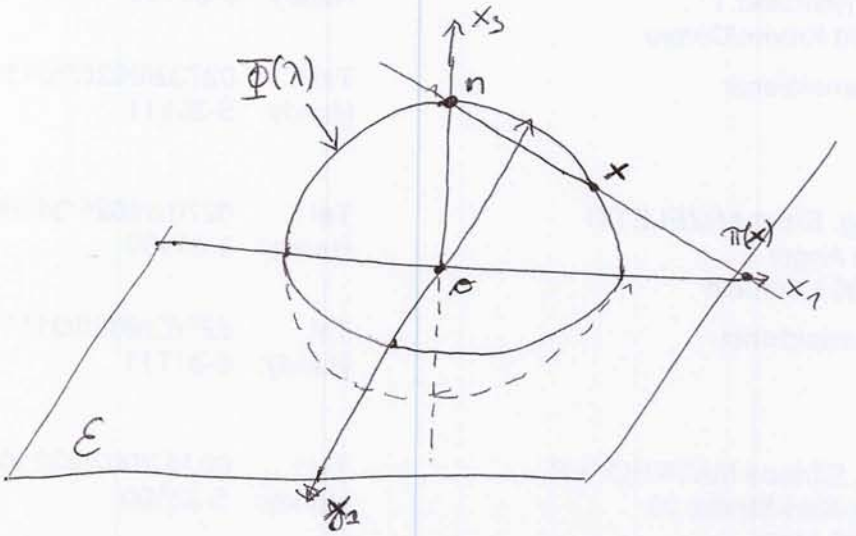
10.3.3

Ellipsoid $\Phi(\lambda)$ mit $\lambda\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = x_1^2 + x_2^2 + x_3^2 - 1$

$n = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \Phi(\lambda)$

$\mathcal{E} = x_3 = 0$

a) $\pi : \Phi(\lambda) \setminus \{n\} \to \mathcal{E} : x \longmapsto (n \vee x) \cap \mathcal{E}$



π wohldefiniert, da nur die Tangente durch n $\mathcal{E}$ nie schneidet, die ist aber ausgenommen.

Zu jedem Punkt $y \in \mathcal{E}$ $\exists^{\times}$ Verbindungsgerade mit n, diese schneidet $\Phi(\lambda)$ in genau einem Punkt. $\Rightarrow$ π bijektiv.

b) $n \vee x = n + [x - n] = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ x_3 - 1 \end{pmatrix} \cdot K = \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix}$

LGS:

$x_1 \cdot K = y_1$

$x_2 \cdot K = y_2$

$(x_3 - 1) \cdot K = -1 \Rightarrow K = \frac{1}{1 - x_3}$

$\Rightarrow \pi(x) = \begin{pmatrix} \frac{x_1}{1 - x_3} \\ \frac{x_2}{1 - x_3} \\ 0 \end{pmatrix} \qquad x_3 \in [-1, 1)$

$$\pi^{-1}\begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix} = \begin{pmatrix} y_1\,(1-x_3) \\ y_2\,(1-x_3) \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (1-\overset{?}{x_3})\begin{pmatrix} y_1 \\ y_2 \\ -1 \end{pmatrix}$$

$$\pi^{-1}(y) \in \Phi(\lambda)$$

$$\Rightarrow y_1^2\,(1-x_3)^{\cancel{2}} + y_2^2\,(1-x_3)^{\cancel{2}} = 1 - x_3^2 = (1+x_3)\cancel{(1-x_3)}$$

$$\Leftrightarrow \quad (1-x_3)\,(y_1^2+y_2^2) = 1+x_3$$

$$\Leftrightarrow \quad \frac{1+x_3}{1-x_3} = \frac{-(1-x_3)+2}{1-x_3} = -1 + \frac{2}{1-x_3} = \frac{\cancel{\ldots}}{y_1^2+y_2^2}$$

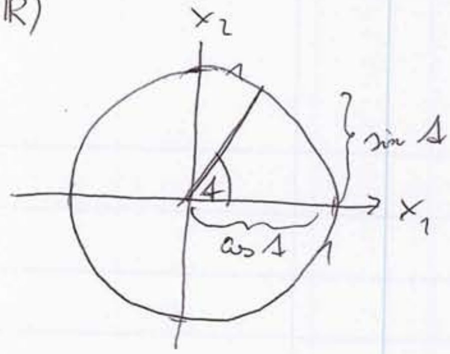
$$\Leftrightarrow \quad 1-x_3 = \frac{2}{y_1^2+y_2^2+1}$$

$$\Rightarrow \pi^{-1}(y) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{2}{y_1^2+y_2^2+1}\begin{pmatrix} y_1 \\ y_2 \\ -1 \end{pmatrix}$$

10.3.4 c)

$\psi: \mathbb{R} \to \mathbb{R}^{2\times 1}: \; t \mapsto \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

$\psi(\mathbb{R})$



Gleichung: $\cos^2 t + \sin^2 t = 1$

$\iff \quad x_1^2 + x_2^2 = 1$

$\Phi(x) = \{x \in \mathbb{R}^{2\times 1} \mid x_1^2 + x_2^2 = 1\}$ ... Ellipse

d) $\psi: \mathbb{R} \to \mathbb{R}^{2\times 1}: \; t \mapsto \begin{pmatrix} t \\ \frac{1}{t} \end{pmatrix}$

$\Rightarrow x_2 = \frac{1}{x_1} \iff x_1 x_2 - 1 = 0$

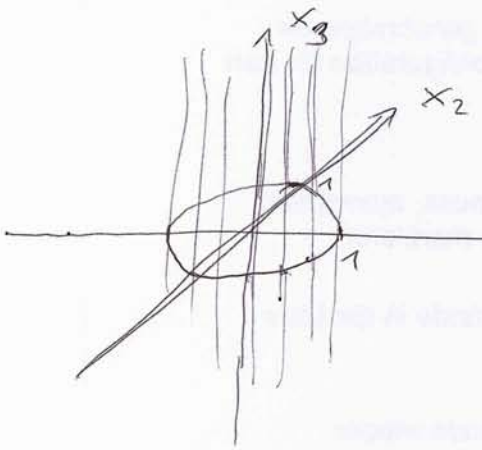
Einbettung und Koo.-Transf.

$$\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{array} \overset{S}{\mapsto}\overset{Z}{\mapsto} \begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{array} \overset{S}{\mapsto} \begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{4} \end{array}$$

$$\overset{Z}{\mapsto} \begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{4} \end{array} \begin{matrix} \\ \\ \cdot 2 \end{matrix} \overset{S}{\mapsto}\overset{Z}{\mapsto} \begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}$$

$\Rightarrow$ Gl.: $\underline{x_1^2 - x_2^2 = 1}$ ... Hyperbel

h) $\psi: \mathbb{R}^2 \to \mathbb{R}^{3\times 1}: \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos A_1 \\ \sin A_1 \\ A_2 \end{pmatrix}$



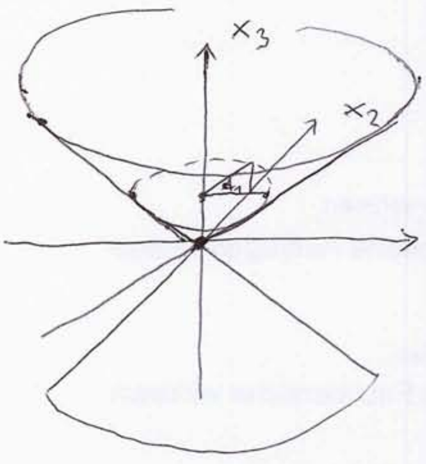
Ellipse in Ebene (x_1, x_2)

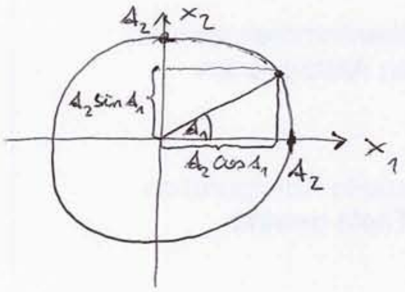
x_3 unabh.

Gl.: $x_1^2 + x_2^2 = 1 \ldots$

$\Rightarrow$ elliptischer Zylinder

k) $\psi: \mathbb{R}^2 \to \mathbb{R}^{3\times 1}: (A_1, A_2) \mapsto \begin{pmatrix} A_2 \cos A_1 \\ A_2 \sin A_1 \\ A_2 \end{pmatrix}$





$(A_2 \sin A_1)^2 + (A_2 \cos A_1)^2 = A_2^2$

$\Leftrightarrow \quad \sin^2 A_1 + \cos^2 A_1 = 1$

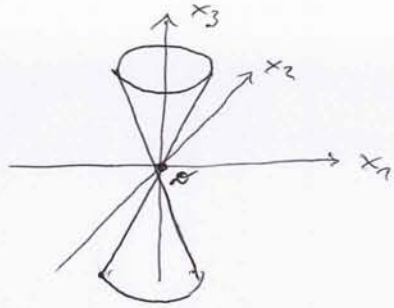
$\Rightarrow \quad \left(\frac{x_2}{x_3}\right)^2 + \left(\frac{x_1}{x_3}\right)^2 = 1$

$\Leftrightarrow \quad x_1^2 + x_2^2 = x_3^2$

$\Leftrightarrow \quad x_1^2 + x_2^2 - x_3^2 = 0 \ldots$ Kegel

10.3.5

Kegel $\Phi(\lambda)$ $\quad \lambda\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 + x_2^2 - x_3^2$



a)

1. Hyperbel

$\mathcal{E}: x_1 = 1$

$\Rightarrow \mathcal{E} \cap \Phi: \; 1 + x_2^2 - x_3^2 = 0 \iff x_2^2 - x_3^2 = -1$ ... Hyperbel

2. Parabel

$\mathcal{E}: \; x_1 - x_3 = 1$

$\iff x_3 = x_1 - 1$

$\Rightarrow \mathcal{E} \cap \Phi: \; x_1^2 + x_2^2 - (x_1 - 1)^2 = \cancel{x_1^2} + x_2^2 - \cancel{x_1^2} + 2x_1 - 1 = 0$

$\iff x_2^2 = -2x_1 + 1$ ... Parabel

3. Ellipse

$\mathcal{E}: \; x_3 = -1$

$\Rightarrow \mathcal{E} \cap \Phi: \; x_1^2 + x_2^2 - 1 = 0 \iff x_1^2 + x_2^2 = 1$ ... Ellipse (Kreis)

b) 1. $\mathcal{E}: x_1 = 1 \Rightarrow \mathcal{E}': x_1 = 0$

$\Rightarrow \mathcal{E}' \cap \Phi: \begin{cases} x_1 = 0 \\ x_2^2 - x_3^2 = 0 \iff (x_2 - x_3)(x_2 + x_3) = 0 \end{cases}$

2 Geraden

2. $\mathcal{E}: x_1 - x_3 = 1 \Rightarrow \mathcal{E}': \; x_1 - x_3 = 0 \iff x_1 = x_3$

$\Rightarrow \mathcal{E}' \cap \Phi: \begin{cases} x_2^2 = 0 \iff x_2 = 0 \\ x_1 - x_3 = 0 \end{cases}$

Gerade

3. $\mathcal{E}: x_3 = -1 \Rightarrow \mathcal{E}': \; x_3 = 0$

$\Rightarrow \mathcal{E}' \cap \Phi: \begin{cases} x_3 = 0 \\ x_1^2 + x_2^2 = 0 \end{cases}$, also nur o