Lin. Algebra UE

V) 9.8: 2α
9.9: 1αβ
9.10: 1, 5, 6, 7αβ, 9, 12

9.8.2α) Transformation in Normalform $\operatorname{diag}(\square, \dots, \square, 0, \dots, 0)$ mit $\square = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$

$$\begin{array}{cccc} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ \hline 0&1&2&1 \\ 2&0&1&1 \\ 1&2&0&1 \\ 2&2&2&0 \end{array} \overset{s}{\mapsto} \begin{array}{cccc} 1&0&0&0 \\ 0&1&1&2 \\ 0&0&1&0 \\ 0&0&0&1 \\ \hline 0&1&0&0 \\ 2&0&1&1 \\ 1&2&2&2 \\ 2&2&1&1 \end{array} \overset{z}{\mapsto} \begin{array}{cccc} &&& \\ \hline 0&1&0&0 \\ 2&0&1&1 \\ 0&2&0&0 \\ 0&2&0&0 \end{array} \overset{s}{\mapsto} \overset{z}{\mapsto} \begin{array}{cccc} 0&1&0&0 \\ 1&0&1&2 \\ 0&0&1&0 \\ 0&0&0&1 \\ \hline \boxed{\begin{matrix} 0&2 \\ 1&0 \end{matrix}}&4&1 \\ &&0&0 \\ 2&0&0&0 \\ 2&0&0&0 \end{array}$$

$A := \begin{pmatrix} 0&1&2&1 \\ 2&0&1&1 \\ 1&2&0&1 \\ 2&2&2&0 \end{pmatrix}$

$$\overset{s}{\mapsto} \overset{z}{\mapsto} \begin{array}{cccc} 0&1&1&1 \\ 1&0&1&2 \\ 0&0&1&0 \\ 0&0&0&1 \\ \hline 0&2&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{array} \begin{array}{l} \left.\vphantom{\begin{matrix}0\\0\\0\\0\end{matrix}}\right\} =: P \\ \left.\vphantom{\begin{matrix}0\\0\\0\\0\end{matrix}}\right\} \operatorname{diag}(\square, 0, 0) \end{array}$$

$\underline{\operatorname{diag}(\square, 0, 0) = P^T \cdot A \cdot P}$

9.9.1α)

$$\begin{array}{ccc} 1&0&0 \\ 0&1&0 \\ 0&0&1 \\ \hline -5&1&-1 \\ 1&-2&2 \\ -1&2&-2 \end{array} \overset{s}{\mapsto} \begin{array}{ccc} 1&0&0 \\ 0{,}5&1&1 \\ 0&0&1 \\ \hline -4{,}5&1&0 \\ 0&-2&0 \\ 0&2&0 \end{array} \overset{z}{\mapsto} \begin{array}{ccc} && \\ \hline -4{,}5&0&0 \\ 0&-2&0 \\ 0&0&0 \end{array} \overset{s}{\mapsto} \overset{z}{\mapsto} \begin{array}{ccc} -\frac{\sqrt{2}}{3}i&0&0 \\ -\frac{1}{\sqrt{2}3}i&-\frac{1}{\sqrt{2}}i&1 \\ 0&0&1 \\ \hline 1&0&0 \\ 0&1&0 \\ 0&0&0 \end{array}$$

β)

$$\begin{array}{ccc} 1&0&0 \\ 0&1&0 \\ 0&0&1 \\ \hline 0&1&1 \\ 1&0&1 \\ 1&1&0 \end{array} \overset{s}{\mapsto} \begin{array}{ccc} 1&0&0 \\ 0&1&0 \\ 1&0&1 \\ \hline 1&1&1 \\ 2&0&1 \\ 1&1&0 \end{array} \overset{z}{\mapsto} \begin{array}{ccc} && \\ \hline 2&2&1 \\ 2&0&1 \\ 1&1&0 \end{array} \overset{s}{\mapsto} \begin{array}{ccc} 1&-1&-0{,}5 \\ 0&1&0 \\ 1&-1&0{,}5 \\ \hline 2&0&0 \\ 2&-2&0 \\ 1&0&-0{,}5 \end{array} \overset{z}{\mapsto} \begin{array}{ccc} && \\ \hline 2&0&0 \\ 0&-2&0 \\ 0&0&-0{,}5 \end{array}$$

$$\overset{s}{\mapsto} \overset{z}{\mapsto} \frac{P}{E_3} \qquad P := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & i \\ 0 & -i & 0 \\ 1 & i & -i \end{pmatrix}$$

9.10.1) $(\alpha) \sim \operatorname{diag}(-1, -1, 0)$ ⇒ neg. semidef.

$(\beta) \sim \operatorname{diag}(1, -1, -1)$ ⇒ indef.

(δ) indef., da $e_1^T \cdot (\delta) \cdot e_1 = -1$, aber $e_2^T \cdot (\delta) \cdot e_2 = +1$

9.10.5) $G = \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & 0 \\ i & 0 & 2 \end{pmatrix}$

a) $\left.\begin{array}{l} \det G_{(1)} = 1 \\ \det G_{(1,2)} = 1 \\ \det G_{(1,2,3)} = 1 \end{array}\right\} \Rightarrow G$ pos. def.

b)
$$\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & -i \\ 0 & 1 & 0 \\ i & 0 & 2 \end{array} \overset{S}{\mapsto} \begin{array}{ccc} 1 & 0 & +i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ i & 0 & 1 \end{array} \overset{Z}{\mapsto} \begin{array}{ccc} 1 & 0 & +i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}$$

$\left.\begin{array}{ccc} 1 & 0 & +i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right\} =: Q$

$$\overline{Q}^T \cdot G \cdot Q = E_3 \Leftrightarrow G = \underbrace{(\overline{Q}^T)^{-1}}_{=(\overline{Q}^{-1})^T = (\overline{Q^{-1}})^T} \cdot E_3 \cdot \underbrace{Q^{-1}}_{=: P} \Leftrightarrow G = \overline{P}^T \cdot P$$

$$\Rightarrow P = \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

9.10.7) α) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto a x_1^2 + 2a x_1 x_2 + 2(a+1) x_2^2 + 2 x_2 x_3 + x_3^2$

$$\begin{pmatrix} a & a & 0 \\ a & 2(a+1) & 1 \\ 0 & 1 & 1 \end{pmatrix} \overset{S}{\mapsto} \begin{pmatrix} a & 0 & 0 \\ a & a+2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \overset{Z}{\mapsto} \begin{pmatrix} a & 0 & 0 \\ 0 & a+2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \overset{S}{\mapsto} \overset{Z}{\mapsto} \begin{pmatrix} a & & \\ & a+1 & \\ & & 1 \end{pmatrix}$$

pos. def.: $a > 0$

$a = 0$: pos. semidef
$a < 0$: indef.

β) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto -a x_1^2 - 2a x_1 x_2 + 2 x_2 x_3 + x_3^2$

$$\begin{pmatrix} -a & -a & 0 \\ -a & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \overset{S}{\mapsto} \begin{pmatrix} -a & 0 & 0 \\ -a & a & 1 \\ 0 & 1 & 1 \end{pmatrix} \overset{Z}{\mapsto} \begin{pmatrix} -a & 0 & 0 \\ 0 & a & 1 \\ 0 & 1 & 1 \end{pmatrix} \overset{S}{\mapsto} \overset{Z}{\mapsto} \begin{pmatrix} -a & & \\ & a-1 & \\ & & 1 \end{pmatrix}$$

pos. def.: $\emptyset$

$\forall a \in \mathbb{R}$ indef.

9.10.9)

$$\sigma(E,E) = \begin{pmatrix} -2 & i & -1 & 0 \\ -i & -2 & 0 & i \\ -1 & 0 & -1 & 0 \\ 0 & -i & 0 & -2 \end{pmatrix} =: G$$

a) $\det G_{(1)} = -2$

$\det G_{(1,2)} = 3$

$$\det G_{(1,2,3)} = \begin{vmatrix} -2 & i & -1 \\ -i & -2 & 0 \\ -1 & 0 & -1 \end{vmatrix} = -1$$

$$\det G_4 = \begin{vmatrix} -2 & i & -1 & 0 \\ -i & -2 & 0 & i \\ -1 & 0 & -1 & 0 \\ 0 & -i & 0 & -2 \end{vmatrix} = (-1)\begin{vmatrix} i & -1 & 0 \\ -2 & 0 & i \\ -i & 0 & -2 \end{vmatrix} - 1 \begin{vmatrix} -2 & i & 0 \\ -i & -2 & i \\ 0 & -i & -2 \end{vmatrix} = -3 + 4 = 1$$

$\Rightarrow G$ neg. def.

b) ges.: Gleichung von $\begin{pmatrix} i \\ 1 \\ i \\ 1 \end{pmatrix}^{\perp_\sigma}$

$$(-i, 1, -i, 1) \cdot \sigma(E,E) = (2i, -1-i, 2i, i-2)$$

$$\Rightarrow 2i\,x_1 - (1+i)\,x_2 + 2i\,x_3 + (i-2)\,x_4 = 0$$

9.10.12) $A = (a_{ij}) \in \mathbb{K}^{n\times n}$ ($\mathbb{K} = \mathbb{R}$ bzw. $\mathbb{K} = \mathbb{C}$) pos. def. symm. bzw. hermitesche Matrix

a) ZZ: $|a_{ij}| < \max\{a_{ii}, a_{jj}\} \quad \forall i,j \in \{1,\dots,n\}$ mit $i \neq j$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix}$$

Fall $\mathbb{R}$: $a_{ij} = a_{ji}$

Fall $\mathbb{C}$: $a_{ij} = \overline{a_{ji}}$ $\quad \forall i,j$

Fall $\mathbb{R}$: A pos. def. $\Rightarrow a_{ii} > 0 \ \forall i$ sowie $\det A_{(p)} > 0$

$$\det A_{(1,2)} = a_{11}a_{22} - a_{12}^2 > 0$$
$$\Leftrightarrow (\max\{a_{11}, a_{22}\})^2 \geq a_{11}a_{22} > a_{12}^2$$
$$\Leftrightarrow \max\{a_{11}, a_{22}\} > |a_{12}|$$

allgemein:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots a_{1i} & \cdots a_{1j} \cdots \\ a_{21} & a_{22} & & \\ \vdots & & & \\ a_{i1} & \cdots & a_{ii} & \\ \vdots & & & \\ a_{j1} & \cdots & & a_{jj} \cdots \\ \vdots & & & \end{bmatrix} \overset{S}{\longmapsto} \begin{bmatrix} a_{1i} & a_{1j} & \cdots \\ a_{2i} & a_{2j} & \cdots \\ \vdots & & \\ a_{ii} & a_{ij} & \cdots \\ \vdots & & \\ a_{ji} & a_{jj} & \cdots \\ \vdots & & \ddots \end{bmatrix} \overset{Z}{\longmapsto} \begin{bmatrix} a_{ii} & a_{ij} & \cdots \\ a_{ji} & a_{jj} & \cdots \\ \vdots & & \ddots \end{bmatrix}$$

kongruente Matrix auch pos. def. $\Rightarrow$ weiter wie oben

Fall $\mathbb{C}$: $a_{ij} = \overline{a_{ji}} \ \forall i,j \Rightarrow a_{ii} \in \mathbb{R} \ \forall i$

$$|a_{ij}| = \sqrt{a_{ij} \cdot a_{ji}} \in \mathbb{R} \ \forall i,j$$

$\Rightarrow$ Beweis analog zu Fall $\mathbb{R}$!

b) ZZ: $\exists k \in \{1, \dots, n\}$ mit $|a_{ij}| \leq a_{kk} \quad \forall i,j$

$$|a_{ij}| \leq \max\{|a_{ij}|\} =: a_{mn} \overset{a)}{\leq} \max\{a_{mm}, a_{nn}\} = a_{kk} \quad \forall i,j$$