25.03.09

Lin. Algebra UE

III, 8.7: 11
8.8: 4, 5
8.9: 1β, 2
9.2: 1βε~~δ~~ϑζ, 2
9.3: 1

8.7.11) $\dim V = n < \infty$

·) n versch. EW $\Rightarrow$ f einfach strukt. lt. Satz 8.5.5

·) sonst $\exists$ mind. ein $J_*(\lambda_i)$-Kästchen mit $\circledast \geq 2$ (für eine Matrix $\langle B^*, f(B)\rangle$ in JNF)

lt. Hinweis: betrachten nur die linke obere 2×2 Ecke des Kästchens $J_*(\lambda_i) =: A_{\lambda_i}$

$$A_{\lambda_i} = \begin{pmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{pmatrix} \neq id_V \quad \forall \lambda_i \in K$$

$$\Rightarrow A_{\lambda_i}^k = \begin{pmatrix} \lambda_i^k & k\cdot\lambda_i^{k-1} \\ 0 & \lambda_i^k \end{pmatrix} \neq id_V \quad \forall k \in \mathbb{N}^\times,\ \lambda_i \in K$$

$\Rightarrow f^k = id_V$ kann nur bei 1×1 Kästchen gelten $\Rightarrow \langle B^*, f(B)\rangle = diag(\lambda_1, \dots, \lambda_n)$

$\Rightarrow$ f einfach strukt. lt. Def.

8.8.4) $(V, \mathbb{C})$ mit $\dim(V, \mathbb{C}) = n$

enthält Vektoren der Form $\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} a_1 + i b_1 \\ \vdots \\ a_n + i b_n \end{pmatrix}$ mit $(a, b) \in \mathbb{R}\times\mathbb{R}$

<u>$\dim(V, \mathbb{C}) = 1$</u>

Basis: $(\underset{\neq 0}{z}) = (a + ib) \overset{A\,8.11\,b)}{\Rightarrow}$ Basis $(V, \mathbb{R})$: $\left\{ \overset{b_1}{\begin{pmatrix} a \\ b \end{pmatrix}}, \overset{b_2}{\begin{pmatrix} -b \\ a \end{pmatrix}} \right\}$ mit $(a, b) \neq (0,0) \in \mathbb{R}\times\mathbb{R}$

also $\dim(V, \mathbb{R}) = 2$

Gaußsche Zahlenebene:
$z = 1 \Rightarrow a = 1,\ b = 0$
$\Rightarrow b_1 = (1, 0)^T,\ b_2 = (0, 1)^T$

<u>$\dim(V, \mathbb{C}) = n$</u>

Basis: $\left\{ \begin{pmatrix} z_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ z_n \end{pmatrix} \right\}$

$\Rightarrow$ Basis $(V, \mathbb{R})$: $\left\{ \begin{pmatrix} a_1 \\ b_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -b_1 \\ a_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ a_2 \\ b_2 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -b_2 \\ a_2 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ a_n \\ b_n \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ -b_n \\ a_n \end{pmatrix} \right\}$

also $\dim(V, \mathbb{R}) = 2n$

ev. noch Isomorphie zeigen

8.8.5)

$$f_c\begin{pmatrix}1\\1\\-1\end{pmatrix}=0 \qquad f_c\begin{pmatrix}0\\1\\i\end{pmatrix}=i\begin{pmatrix}0\\1\\i\end{pmatrix}=\begin{pmatrix}0\\i\\-1\end{pmatrix} \qquad f_c\begin{pmatrix}0\\1\\-i\end{pmatrix}=-i\begin{pmatrix}0\\1\\-i\end{pmatrix}=\begin{pmatrix}0\\-i\\-1\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&i&0\\0&0&-i\end{pmatrix} = P^{-1}\cdot A_{f_c}\cdot \underbrace{\begin{pmatrix}1&0&0\\1&1&1\\-1&i&-i\end{pmatrix}}_{=:P}$$

$$A_{f_c} = \begin{pmatrix}1&0&0\\1&1&1\\-1&i&-i\end{pmatrix}\begin{pmatrix}0&0&0\\0&i&0\\0&0&-i\end{pmatrix}P^{-1}$$

$$= \begin{pmatrix}0&0&0\\0&i&-i\\0&-1&-1\end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix}2&0&0\\-1-i&1&-i\\-1+i&1&i\end{pmatrix} = \begin{pmatrix}0&0&0\\1&0&1\\1&-1&0\end{pmatrix}$$

$$A_{f_c} = A_f = \langle E^*, f(E)\rangle$$

$$\chi_f(x) = \cancel{\det(A_f - XE_n)} = \det(f - X\,id_V) = \begin{vmatrix}-x&0&0\\1&-x&1\\1&-1&-x\end{vmatrix} = (-x)^3 - x$$

$$= -x\,(x^2+1)$$

$$\Rightarrow \chi_{f_c}(x) = -x\,(x+i)(x-i)$$

8.9.1, B)

$$\langle B^*, f(B)\rangle = \begin{pmatrix}2&1&-1&0\\0&1&-1&0\\2&1&-1&0\\0&2&-1&-1\end{pmatrix}$$

a) $$\chi_f(x) = \det(\langle B^*, f(B)\rangle - XE_n) = \det\begin{pmatrix}2-x&1&-1&0\\0&1-x&-1&0\\2&1&-1-x&0\\0&2&-1&-1-x\end{pmatrix}$$

$$= (-1-x)\begin{vmatrix}2-x&1&-1\\0&1-x&-1\\2&1&-1-x\end{vmatrix} = (-1-x)\big((2-x)(1-x)(-1-x)-2+2(1-x)+(2-x)\big)$$

$$= (-1-x)\big((2-x)(x^2-1)-3x+2\big) = (-1-x)(-x^3+2x^2-2x)$$

$$= x\,(x+1)\,(x^2-2x+2)$$

$$\chi_{f_c}(x) = x\,(x+1)\,(x-(1+i))(x-(1-i))$$

b)

$$\langle B^*, f(B)\rangle \approx \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1+i & 0 \\ 0 & 0 & 0 & 1-i \end{pmatrix} \mathrel{\hat=} \text{Matrix der EW} =: \langle C^*, f(C)\rangle$$

$$\Rightarrow \langle B^*, f(B)\rangle \approx \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} =: \langle D^*, f(D)\rangle$$

c) $\langle D^*, f(D)\rangle = \langle D^*, \overset{C}{B}\rangle \langle \overset{C}{B}{}^*, f(\overset{C}{B})\rangle \langle \overset{C}{B}{}^*, D\rangle$

$$= \langle D^*, \overset{C}{B}\rangle \langle \overset{C}{B}{}^*, B\rangle \langle B^*, f(B)\rangle \underbrace{\langle B^*, \overset{C}{B}\rangle \langle \overset{C}{B}{}^*, D\rangle}_{\cancel{\langle B^*, D\rangle = D}}$$

Berechne Eigenvektoren:

$\ker(A - 0E_n)$:

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline 2 & 1 & -1 & 0 & \\ 0 & 1 & -1 & 0 & 0 \\ 2 & 1 & -1 & 0 & \\ 0 & 2 & -1 & -1 & \\ \hline 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & \\ 0 & 0 & 0 & 0 & \end{array}$$

$\Rightarrow$ Basis v. ER(0): $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} =: c_1$

$\ker(A + E_n)$:

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline 3 & 1 & -1 & 0 & \\ 0 & 2 & -1 & 0 & 0 \\ 2 & 1 & 0 & 0 & \\ 0 & 2 & -1 & 0 & \end{array}$$

$\Rightarrow$ Basis: $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} =: c_2$

$\ker(A - (1+i)E_n)$:

$$\begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline 1-i & 1 & -1 & 0 & \\ 0 & -i & -1 & 0 & 0 \\ 2 & 1 & -2-i & 0 & \\ 0 & 2 & -1 & -2+i & \\ \hline 1 & 0 & 0 & i & \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & i & \\ 0 & 0 & 0 & 0 & \end{array}$$

$\Rightarrow$ Basis: $\begin{pmatrix} -i \\ 1 \\ -i \\ 1 \end{pmatrix} =: c_3$

$\ker(A - (1-i)E_n) \Rightarrow$ Basis: $\overline{\begin{pmatrix} -i \\ 1 \\ -i \\ 1 \end{pmatrix}} = \begin{pmatrix} i \\ 1 \\ i \\ 1 \end{pmatrix} =: c_4$

$$\Rightarrow \cancel{\langle E^*, B\rangle}\ D = \Big(c_1, c_2, \overbrace{i(c_4 - c_3), c_3 + c_4}^{\text{S.149}}\Big) = \left(\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix} \right)$$

9.2.1) β) $\sigma: \mathbb{R}\times\mathbb{R} \to \mathbb{R}: (x,y) \mapsto xy+1$

$\sigma(x+y,z) = (x+y)z+1 = xz+yz+1$

$\sigma(x,z)+\sigma(y,z) = xz+1+yz+1$ ↯ $\Rightarrow$ keine SLF

ε) $\sigma: \mathbb{C}\times\mathbb{C} \to \mathbb{C}: (x,y) \mapsto \bar{x}x+\bar{x}y$

$\sigma(x+y,z) = (\overline{x+y})(x+y) + (\overline{x+y})z = \bar{x}x+\bar{y}x+\cancel{\bar{y}x}\,\bar{x}y+\bar{y}y+\bar{x}z+\bar{y}z$

$\sigma(x,z)+\sigma(y,z) = \bar{x}x+\bar{x}z+\bar{y}y+\bar{y}z$

$\Rightarrow$ keine SLF, da $\bar{y}x+\bar{x}y \neq 0$

ζ) $\sigma: \mathbb{R}^{2\times1}\times\mathbb{R}^{2\times1} \to \mathbb{R}: \left(\begin{pmatrix}x_1\\x_2\end{pmatrix},\begin{pmatrix}y_1\\y_2\end{pmatrix}\right) \mapsto x_1y_1+y_2$

$\sigma\left(\begin{pmatrix}x_1+y_1\\x_2+y_2\end{pmatrix},\begin{pmatrix}z_1\\z_2\end{pmatrix}\right) = (x_1+y_1)z_1+z_2 = x_1z_1+y_1z_1+z_2$

$\sigma\left(\begin{pmatrix}x_1\\x_2\end{pmatrix},\begin{pmatrix}z_1\\z_2\end{pmatrix}\right)+\sigma\left(\begin{pmatrix}y_1\\y_2\end{pmatrix},\begin{pmatrix}z_1\\z_2\end{pmatrix}\right) = x_1z_1+z_2+y_1z_1+z_2$

$\Rightarrow$ keine SLF

ϑ) $\sigma: C^1(\mathbb{R})\times C^1(\mathbb{R}) \to \mathbb{R}: (f,g) \mapsto f(0)\cdot g'(1)$

1. $\sigma(f+g,h) = (f+g)(0)\cdot h'(1) = (f(0)+g(0))\cdot h'(1)$

   $\sigma(f,h)+\sigma(g,h) = f(0)\cdot h'(1)+g(0)\cdot h'(1) = (f(0)+g(0))\cdot h'(1)$

2. $\sigma(f,g+h) = f(0)\cdot(g+h)'(1) = f(0)\cdot(g'(1)+h'(1))$

   $\sigma(f,g)+\sigma(f,h) = f(0)\cdot g'(1)+f(0)\cdot h'(1) = f(0)\cdot(g'(1)+h'(1))$

3. $\sigma(cf,g) = (cf)(0)\cdot g'(1) = c\cdot f(0)\cdot g'(1)$

   $\varsigma(c)\cdot\sigma(f,g) = \varsigma(c)\cdot f(0)\cdot g'(1)$

4. $\sigma(f,cg) = c\,\sigma(f,g)$, da Mult. kommutativ u. assoziativ

$\Rightarrow$ SLF für $\varsigma = id_{\mathbb{R}}$, also Bilinearform

$d_\sigma: V \to V^*: f \mapsto \sigma(f,\cdot) = f(0)\cdot\cdot'(1)$

$\Rightarrow \ker d_\sigma = \{f\in C^1(\mathbb{R}) \mid f(0)=0\}$

9.3.1) a)

$$f: K^{2\times 2} \to K^{2\times 2} : X \mapsto \square \cdot X^T$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} -b & -d \\ a & c \end{pmatrix} \quad \Rightarrow f \text{ bij.}$$

b) $A \equiv B \Leftrightarrow \exists P \in GL_2(K)$ mit $\varsigma(P^T) \cdot A \cdot P = B$
und $\varsigma \in Aut(K)$

Hinweis: $P^{\#} = (\det P) P^{-1}$

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow P^{\#} = \cancel{(ad-bc)} \cdot \frac{1}{\cancel{ad-bc}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\square \cdot B^T = P^{\#} \cdot (\square \cdot A^T) \cdot P$$

$$\Leftrightarrow \quad f(B) = P^{\#} \cdot f(A) \cdot P$$

$$\Leftrightarrow \square\square f(B) = \square\square P^{\#} \cdot f(A) \cdot P \qquad | \cdot \square^2$$

$$\Leftrightarrow \square\square f(B) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \cdot f(A) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad | \square^2 = \mathrm{diag}(-1,-1)$$

$$\Leftrightarrow f^{-1}(\square) f^{-1}(\square) f^{-1}(f(B)) = (-1)\begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \cdot f^{-1}(f(A)) \cdot \begin{pmatrix} c & -a \\ d & -b \end{pmatrix} \qquad | f^{-1}$$

$$\Leftrightarrow \quad B = \underbrace{\begin{pmatrix} c & d \\ -a & -b \end{pmatrix}}_{= R^T} \cdot A \cdot \underbrace{\begin{pmatrix} c & -a \\ d & -b \end{pmatrix}}_{=: R}$$

c) $M(A_1) \equiv M(A_2) \Leftrightarrow \square \cdot M(A_1)^T = P^{\#}\left(\square \cdot M(A_2)^T\right) \cdot P$

$$\Leftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & A_1 \\ -1 & 0 \end{pmatrix} = P^{\#}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & A_2 \\ -1 & 0 \end{pmatrix}\right) \cdot P$$

$$\Leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix} = (\det P) P^{-1} \begin{pmatrix} 1 & 0 \\ 0 & A_2 \end{pmatrix} \cdot P$$

$$\Leftrightarrow \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix}}_{=: T_1} = P^{-1} \underbrace{\begin{pmatrix} \det P & 0 \\ 0 & \det P A_2 \end{pmatrix}}_{=: T_2} \cdot P$$

$$\chi_{T_1}(x) = \begin{vmatrix} 1-x & 0 \\ 0 & A_1 - x \end{vmatrix} = (1-x)(A_1 - x)$$

$$\chi_{T_2}(x) = \quad (\det P - x)(\det P A_2 - x)$$

$$\Rightarrow T_1 \approx T_2 \Leftrightarrow \chi_{T_1}(x) = \chi_{T_2}(x)$$

$\rightarrow$ 2 Fälle: • $\det P = 1 \Rightarrow \chi_{T_2}(x) = (1-x)(A_2 - x) \Rightarrow \underline{A_1 = A_2}$

• $\det P A_2 = 1$
$\Leftrightarrow \det P = \frac{1}{A_2} \Rightarrow \chi_{T_2}(x) = \left(\frac{1}{A_2} - x\right)(1-x) \Rightarrow \underline{A_1 = \frac{1}{A_2}}$

8.9.2) $\kappa = P(f): P(\mathbb{R}^{3\times 3}) \to P(\mathbb{R}^{3\times 3})$
$\qquad K_x \mapsto K(f(x))$

$\langle B^*, f(B)\rangle$ sei Koo.-Matrix in reeller JNF $\Rightarrow$ jede Abbildungsmatrix ist zu genau einer dieser Formen ähnlich:

fällt weg, da $= P(id_{\mathbb{R}})$ $\cancel{\begin{pmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & \lambda_1 \end{pmatrix}}$ $\begin{pmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & \lambda_2 \end{pmatrix}$ $\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$

$\begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & 1 \\ & \lambda_1 \end{matrix}} & \\ & \lambda_1 \end{pmatrix}$ $\begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & 1 \\ & \lambda_1 \end{matrix}} & \\ & \lambda_2 \end{pmatrix}$ $\begin{pmatrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{pmatrix}$ $\begin{pmatrix} \lambda_1 & & \\ & a & -b \\ & b & a \end{pmatrix}$

a) Q Fixpunkt $\Leftrightarrow \kappa(Q) = Q \Leftrightarrow f(q) = \lambda \cdot q$

b) $Q_1 \vee Q_2$ Fixgerade $\Leftrightarrow \kappa(Q_1 \vee Q_2) = Q_1 \vee Q_2 \Leftrightarrow f(x) \in f([q_1, q_2]) = \lambda_1 q_1 + \lambda_2 q_2 \quad \forall x \in [q_1, q_2]$

c) $Q_1 \vee Q_2$ Fixpunktgerade $\Leftrightarrow \kappa(Q) = Q \;\; \forall Q \in Q_1 \vee Q_2 \Leftrightarrow f(q) = q \;\; \forall q \in [q_1, q_2]$

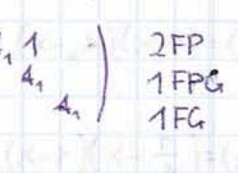
d) alle Geraden durch Q_i sind fix

betrachte zB: $\begin{pmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & \lambda_2 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} \lambda_1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \lambda_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \lambda_2 \end{pmatrix} \right\}$ sind Fixpunkte

$\quad Q_1 \;\; Q_2 \;\; Q_3$

$H\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \\ 0 & 0 \end{pmatrix}$ ist Fixpunktgerade

$H\begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \\ 0 & \lambda_2 \end{pmatrix}, H\begin{pmatrix} 0 & 0 \\ \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ sind Fixgeraden



(2)

analog: $\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$ 3 FP, 3 FG $\Rightarrow$ (1)

$\begin{pmatrix} \lambda_1 & 1 & \\ & \lambda_1 & \\ & & \lambda_1 \end{pmatrix}$ 2 FP, 1 FPG, 1 FG $\Rightarrow$ (4)

$\begin{pmatrix} \lambda_1 & 1 & \\ & \lambda_1 & \\ & & \lambda_2 \end{pmatrix}$ 2 FP, 2 FG $\Rightarrow$ (3)

$\begin{pmatrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{pmatrix}$ 1 FP, 1 FG $\Rightarrow$ (6)

$\begin{pmatrix} \lambda_1 & & \\ & a & -b \\ & b & a \end{pmatrix}$ 1 FP, 1 FG $\not\ni Q_1$ $\Rightarrow$ (5)