4.6.09

Wahrscheinlichkeitsrechnung u. Statistik UE

63) a) $\mathbb{E}X = 3$, $\mathbb{E}X^2 = 10$, $\mathbb{E}Y = 2$, $\mathbb{E}Y^2 = 29$, $\mathbb{E}XY = 0$

$$\rho_{XY} = \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{var} X}\sqrt{\operatorname{var} Y}} = \frac{\mathbb{E}XY - \mathbb{E}X\,\mathbb{E}Y}{\sqrt{\mathbb{E}X^2 - (\mathbb{E}X)^2}\sqrt{\mathbb{E}Y^2 - (\mathbb{E}Y)^2}} = \frac{-6}{1\cdot 5} = -1{,}2$$

$\lightning$ zu $|\rho_{XY}| \leq 1$.

b)

65) $X, Y \sim N(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$; ZZ: $\operatorname{cor}(X,Y) = \rho$

$$\operatorname{cor}(X,Y) = \frac{\mathbb{E}XY - \mu_X \cdot \mu_Y}{\sigma_X \cdot \sigma_Y}$$

$$\mathbb{E}XY = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, y\, f(x,y)\, dx\, dy$$

vgl. S.60 Bsp. 4.3 Def. von $\tilde{x}, \tilde{y}$ übernommen.

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}\,\sigma_X\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}(\tilde{x}-\rho\tilde{y})^2\right] \frac{y}{\sqrt{2\pi}\,\sigma_Y} \exp\left(-\frac{\tilde{y}^2}{2}\right) dx\, dy$$

$$= \int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi}\,\sigma_Y} \exp\left(-\frac{\tilde{y}^2}{2}\right)\left(\int_{-\infty}^{\infty} x \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma_X\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}(\tilde{x}-\rho\tilde{y})^2\right]}_{=: f_Z(x)} dx\right) dy$$

$$-\frac{(\tilde{x}-\rho\tilde{y})^2}{2(1-\rho^2)} = -\frac{1}{2}\frac{\left(\frac{x-\mu_X}{\sigma_X} - \rho\tilde{y}\right)^2}{(1-\rho^2)} = -\frac{1}{2}\frac{(x-(\mu_X + \rho\sigma_X\tilde{y}))^2}{(1-\rho^2)\sigma^2 x}$$

$$\Rightarrow Z \sim N(\mu_X + \rho\sigma_X\tilde{y}, (1-\rho^2)\sigma^2 x)$$

$$\Rightarrow \mathbb{E}Z = \int_{-\infty}^{\infty} x\, f_Z(x) = \mu_X + \rho\sigma_X\tilde{y} = \mu_X + \rho\sigma_X \frac{y-\mu_Y}{\sigma_Y}$$

$$\Rightarrow \mathbb{E}XY = \mu_X \overbrace{\int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi}\,\sigma_Y}\exp\left(-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right) dy}^{\mathbb{E}Y}$$

$$+ \rho\sigma_X\left(\underbrace{\int_{-\infty}^{\infty} y^2 \frac{1}{\sqrt{2\pi}\,\sigma_Y^2} \text{———}\,\text{"}\,\text{———}\, dy}_{\frac{\mathbb{E}Y^2}{\sigma_Y}} - \mu_Y \underbrace{\int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi}\,\sigma_Y^2}\exp\left(-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right) dy}_{\frac{\mathbb{E}Y}{\sigma_Y}}\right)$$

$$= \mathbb{E}X\,\mathbb{E}Y + \frac{\rho\sigma_X}{\cancel{\sigma_Y}}\underbrace{(\mathbb{E}Y^2 - (\mathbb{E}Y)^2)}_{=\sigma_Y^2}$$

$$\Rightarrow \operatorname{cor}(X,Y) = \frac{\cancel{\mu_X\mu_Y} + \rho\cancel{\sigma_X\sigma_Y} - \cancel{\mu_X\mu_Y}}{\cancel{\sigma_X\sigma_Y}} = \rho$$

66) $X_1 \sim B_{n,p}$ $\quad X_1, X_2$ unabh. $\Big\} \Rightarrow X_1 + X_2 \sim B_{n+m,p}$
$X_2 \sim B_{m,p}$

ges.: Verteilung von $X_1 \mid X_1 + X_2 = s$.

$$P[X_1 = i \mid X_1 + X_2 = s] = \frac{P[(X_1=i) \cap (X_1+X_2=s)]}{P[X_1+X_2=s]} \quad \Leftrightarrow P[(X_1=i)\cap(X_2=s-i)]$$

$$\overset{\text{unabh.}}{=} \frac{P[X_1=i]\cdot P[X_2=s-i]}{P[X_1+X_2=s]}$$

$$= \frac{\binom{n}{i} \cancel{p^i (1-p)^{n-i}} \cdot \binom{m}{s-i} \cancel{p^{s-i} (1-p)^{m-s+i}}}{\binom{n+m}{s} \cancel{p^s (1-p)^{n+m-s}}}$$

$$\Rightarrow X_1 \mid X_1 + X_2 = s \sim H_{n+m,n,s}$$

67) $f(x,y) = \begin{cases} \frac{1}{x} & 0 < x \le 1,\ 0 < y \le x \\ 0 & \text{sonst} \end{cases}$

$f_X(x) = 1 \quad 0 < x \le 1$
$f_Y(y) = -\ln y \quad 0 < y \le 1$

ges.: ~~$f(x|y)$~~ Dichten von $X \mid Y=y$, $Y \mid X=x$, zugeh. Erwartungswerte

Dichte von $X \mid Y=y$:

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{1}{x}}{-\ln y} = \begin{cases} -\frac{1}{x \ln y} & 0<x\le 1,\ 0<y\le x \\ 0 & \text{sonst} \end{cases}$$

Dichte von $Y \mid X=x$:

$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \begin{cases} \frac{1}{x} & 0<x\le 1,\ 0<y\le x \\ 0 & \text{sonst} \end{cases}$$

$$\mathbb{E}(X \mid Y=y) = \int_y^1 x\, f(x|y)\, dx = -\int_y^1 \frac{1}{\ln y}\, dx = \frac{y-1}{\ln y}$$

$$\mathbb{E}(Y \mid X=x) = \int_0^x y\, f(y|x)\, dy = \int_0^x y \frac{1}{x} = \frac{1}{x}\,\frac{y^2}{2}\Big|_0^x = \frac{x}{2}$$

68) $X \sim Ex_{\frac{1}{4}}$, $Y \sim Ey_{\frac{1}{7}}$ $\qquad f(t) = \lambda e^{-\lambda t}$

ges.: $P[X+Y > 12]$

$$h(x) = \int_0^x f_X(t)\, f_Y(x-t)\, dt$$
$$= \int_0^x \frac{1}{4} e^{-\frac{1}{4}t} \frac{1}{7} e^{-\frac{1}{7}(x-t)} dt$$
$$= \frac{1}{28} e^{-\frac{x}{7}} \int_0^x e^{-\frac{3}{28}t}\, dt$$
$$= -\frac{1}{3} \cdot e^{-\frac{x}{7}} \left(e^{-\frac{3}{28}t} \Big|_0^x \right)$$
$$= -\frac{1}{3} \cdot e^{-\frac{x}{7}} \left(e^{-\frac{3}{28}x} - 1 \right) = \frac{1}{3}\left(e^{-\frac{x}{7}} - e^{-\frac{x}{4}} \right)$$

$$P[X+Y > 12] = 1 - P[X+Y \leq 12]$$
$$= 1 - \int_0^{12} h(x)\, dx$$
$$= 1 - \frac{1}{3} \int_0^{12} \left(e^{-\frac{x}{7}} - e^{-\frac{x}{4}} \right) dx$$
$$= 1 - \frac{1}{3}\left(-7 e^{-\frac{x}{7}} \Big|_0^{12} + 4 e^{-\frac{x}{4}} \Big|_0^{12} \right)$$
$$= 1 - \frac{1}{3}\left(-7 e^{-\frac{12}{7}} + 4 e^{-3} - 3 \right)$$
$$= \frac{7}{3} e^{-\frac{12}{7}} - \frac{4}{3} e^{-3} \approx 35{,}38\,\%$$

69) Gewicht $X_i \sim N(30, 4)$

$\rho_{X_i X_j} = Cor(X_i, X_j) = 0{,}1 \quad \forall i \neq j$

ges.: Vert. von $S = X_1 + X_2 + \ldots + X_{10}$, $\mathbb{E}S$, $var(S)$.

Additionstheorem der NV $\Rightarrow S \sim N(\tilde{\mu}, \tilde{\sigma}^2)$

$$\tilde{\mu} = \sum_{i=1}^{10} \mu_i = 10 \cdot \mu_i = 300 = \mathbb{E}S.$$

$$\tilde{\sigma}^2 = \sum_{i=1}^{10} \sigma_i^2 + 2 \sum_{i \neq j} \sigma_i \sigma_j \rho_{X_i X_j}$$

$$= 10 \cdot \sigma_i^2 + 2 \binom{10}{2} \sigma_i^2 \rho = 76 = var\, S.$$

70) $N \sim P_\mu$ $\quad N, X_i$ unabh. $\quad P[X=i] = \frac{\lambda^i}{i!} e^{-\lambda}$

$X_i \sim P_\lambda$

$S = X_1 + X_2 + \ldots + X_N$

(i) ges: $P[S=0]$

$$P[S=0] = \sum_{n=0}^{\infty} \underbrace{P[S=0 \mid N=n]}_{P[X_1+X_2+\ldots+X_n=0] = P[X_1=0]\cdot\ldots\cdot P[X_n=0] = (P[X_i=0])^n} \cdot P[N=n]$$

$$= \sum_{n=0}^{\infty} (P[X_i=0])^n \cdot P[N=n]$$

$$= \sum_{n=0}^{\infty} e^{-\lambda n} \cdot \frac{\mu^n}{n!} e^{-\mu}$$

$$= e^{-\mu} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} (e^{-\lambda} \mu)^n$$

$$= e^{-\mu} \cdot \exp(e^{-\lambda}\mu) = \exp(\mu(e^{-\lambda}-1))$$

(ii) ges: $\mathbb{E}S$, $\operatorname{var} S$, $\mathbb{E}S^3$

$\mathbb{E}S = \mathbb{E}N \cdot \mathbb{E}X = \mu \cdot \lambda$ (Waldsche Id.)

$\operatorname{var} S = \mathbb{E}N \cdot \operatorname{var} X + (\mathbb{E}X)^2 \cdot \operatorname{var} N$

$= \mu \cdot \lambda + \lambda^2 \mu = \mu\lambda(1+\lambda)$

$\mathbb{E}S^3 = \psi_S^{(3)}(1) = (\psi_N \circ \psi_X)^{(3)}(1) = \ldots = \mu\lambda^3(1+3\mu+\mu^2)$

$\psi_N(4) = e^{\mu(4-1)}$ $\quad \psi_X(1) = e^{\lambda(4-1)}$