Übungen zu Einführung in die Wahrscheinlichkeitsrechnung und Statistik Lösungen 6. Übung

39.

$$\sum_{k=m}^{N} k \frac{\binom{k-1}{m-1}\binom{N-k}{A-m}}{\binom{N+1}{A+1}} = \sum_{k=m}^{N} m \frac{\binom{k}{m}\binom{N-k}{A-m}}{\binom{N+1}{A+1} \cdot \frac{A+1}{N+1}} = m \frac{N+1}{A+1} \underbrace{\sum_{k=m}^{N} \frac{\binom{k}{m}\binom{N-k}{A-m}}{\binom{N+1}{A+1}}}_{=1}$$

40.

$$\log X \sim N_{\mu,\sigma^2} \Rightarrow \mathbf{P}(X \le x) = \mathbf{P}(\log X \le \log x) = \Phi(\log x)$$

$$\Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2}$$

$$\mathbb{E}X = \int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x = \int_0^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2} = \left| \begin{array}{c} y = \log x \\ \mathrm{d}y = \frac{1}{x}\,\mathrm{d}x \end{array} \right|$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^y e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} \mathrm{d}y = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\left(y-(\mu+\sigma^2)\right)^2 - \sigma^2(2\mu+\sigma^2)}{2\sigma^2}} \mathrm{d}y$$

$$= e^{\mu + \frac{\sigma^2}{2}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y-(\mu+\sigma^2)}{\sigma}\right)^2} \mathrm{d}y}_{=1}$$

da

$(y-\mu)^2 + 2y\sigma^2 = y^2 - 2y\mu + \mu^2 + 2y\sigma^2 = (y-(\mu+\sigma^2))^2 - 2y\sigma^2 - 2\mu\sigma^2 - \sigma^4 + 2y\sigma^2$
$(y-(\mu+\sigma^2))^2 - \sigma^2(2\mu+\sigma^2)$

41.

42. Sei $p_i = \frac{\binom{6}{i}\binom{39}{6-i}}{\binom{45}{6}}$.

$$\mu = \mathbb{E}X = \sum_{\omega\in\Omega} \mathrm{G}(\omega)\mathbf{P}(\omega) = 3,6 \cdot p_3 + 39,7 \cdot p_4 + 1215,4 \cdot p_5 \frac{38}{39} + 59693,4 \cdot p_5 \frac{1}{39} + 676401,3 \cdot p_6 \approx 0,296$$

$$\begin{aligned}\mathrm{var}(X) = \sum_{\omega\in\Omega} (\mathrm{G}(\omega)-\mu)^2\,\mathbf{P}(\omega) &= \mu^2 \cdot p_0 + \mu^2 \cdot p_1 + \mu^2 \cdot p_2 + (3,6-\mu)^2 \cdot p_3 + \\ &(39,7-\mu)^2 \cdot p_4 + (1215,4-\mu)^2 \cdot p_5 \frac{38}{39} + \\ &(59693,4-\mu)^2 \cdot p_5 \frac{1}{39} + (676401,3-\mu)^2 \cdot p_6 \\ &\approx 58839.90\end{aligned}$$

43. Taylor-Reihenentwicklung von h um μ liefert

$$h(X) = h(\mu) + h'(\mu)(X-\mu) + R(X)$$

Zur Approximation lässt man den Restterm weg.
Da der Erwartungswert linear ist, also $\mathbb{E}(aX+b) = a\mathbb{E}X + b$ erhält man

$$\mathbb{E}(h(X)) = h(\mu) + h'(\mu)\mathbb{E}(X-\mu) = h(\mu) + h'(\mu)(\underbrace{\mathbb{E}(X)}_{=\mu} - \mu) = h(\mu)$$

Für die Varianz nimmt man $\mathrm{var}(aX+b) = a^2\mathrm{var}(X)$ und erhält

$$\mathrm{var}(h(X)) = (h'(\mu))^2\mathrm{var}(X-\mu) = (h'(\mu))^2 \underbrace{\mathrm{var}(X)}_{=\sigma^2} = (h'(\mu))^2\sigma^2$$

44.

45. $X \sim G_p$:

$$\psi(t) = \mathbb{E}t^X = \sum_{x=1}^{\infty} t^x p(1-p)^{x-1} = tp\sum_{x=1}^{\infty}(t-tp)^{x-1} = \frac{tp}{1-t+tp}$$

für $t \in (0,1]$, also ist

$$\psi'(t) = \frac{p(1-t+tp)-(p-1)tp}{(1-t+tp)^2} = \frac{p}{(1-t+tp)^2}$$

$$\Rightarrow \mathbb{E}X = \psi'(1) = \frac{1}{p}$$

46.
$$\int_{-\infty}^{\infty} x^k \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\,\mathrm{d}x = \int_{-\infty}^{\infty} \underbrace{x^{k-1}}_{g} \cdot \underbrace{x \cdot \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}}_{f'}\,\mathrm{d}x$$
$$= \underbrace{\left[-x^{k-1}\frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\right]_{-\infty}^{\infty}}_{=0,\text{ da } \lim_{n\to\pm\infty} e^{-x^2}=0} + (k-1)\int_{-\infty}^{\infty} x^{k-2}\frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}\,\mathrm{d}x$$

$$\mathbb{E}X^k = (k-1)\mathbb{E}X^{k-2}$$
$$\mathbb{E}X = \mu = 0 \Rightarrow \mathbb{E}X^k = 0 \text{ für } k \text{ ungerade}$$
$$\mathbb{E}X^2 = \sigma^2 = 1 \Rightarrow \mathbb{E}X^k = (k-1)\cdot(k-3)\cdot\ldots\cdot 5\cdot 3\cdot 1 \text{ für } k \text{ gerade}$$