Differentialgleichungen UE

VIII, 213, 217, 221, 230, 236, 239, 243, 247

213; $\dot{x} = A \cdot x$ $\quad A = \begin{pmatrix} 2 & 13 & 1 \\ -1 & 5 & 1 \\ 3 & 0 & 0 \end{pmatrix}$ $\quad x(0) = c = (1,1,3)$

Char. Polynom:

$p(\lambda) = (-1)(\lambda-3)(\lambda-(2+2i))(\lambda-(2-2i))$

$\Rightarrow$ komplexe LB der äqu. Gl. 3. Ordnung: $\psi_c(t) = (e^{3t}, e^{(2+2i)t}, e^{(2-2i)t})$

$\Rightarrow$ reelle LB: $\psi(t) = (e^{3t}, e^{2t}\cos 2t, e^{2t}\sin 2t)$

$$W_\psi(t) = \begin{pmatrix} \psi(t) \\ \dot{\psi}(t) \\ \ddot{\psi}(t) \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{2t}\cos 2t & e^{2t}\sin 2t \\ 3e^{3t} & 2e^{2t}(\cos 2t - \sin 2t) & 2e^{2t}(\sin 2t + \cos 2t) \\ 9e^{3t} & -8e^{2t}\sin 2t & 8e^{2t}\cos 2t \end{pmatrix}$$

$$\Rightarrow W_\psi(0) = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 2 & 2 \\ 9 & 0 & 8 \end{pmatrix} \quad \Rightarrow W_\psi(0)^{-1} = \frac{1}{10}\begin{pmatrix} 16 & -8 & 2 \\ -6 & 8 & -2 \\ -18 & 9 & -1 \end{pmatrix}$$

$$W_\varphi(0) = \begin{pmatrix} c \\ A \cdot c \\ A^2 \cdot c \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 \\ 18 & 7 & 3 \\ 130 & 20 & 54 \end{pmatrix}$$

$$\Rightarrow R(c) = W_\psi(0)^{-1} \cdot W_\varphi(0) = \frac{1}{10}\begin{pmatrix} 132 & 0 & 132 \\ -122 & 10 & -102 \\ 14 & 25 & -81 \end{pmatrix}$$

$\Rightarrow$ Lösung des AWP: $\varphi(t) = \psi(t) \cdot R(c) = \Big(\frac{66}{5}e^{3t} + e^{2t}\left(-\frac{61}{5}\cos 2t + \frac{7}{5}\sin 2t\right),$

$e^{2t}\left(\cos 2t + \frac{5}{2}\sin 2t\right),$

$\frac{66}{5}e^{3t} + e^{2t}\left(-\frac{51}{5}\cos 2t - \frac{81}{10}\sin 2t\right)\Big)$

217) $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \underbrace{\begin{pmatrix}3 & 4\\ 2 & 5\end{pmatrix}}_{=:A}\begin{pmatrix}x\\ y\end{pmatrix}$

$$\chi_A(\lambda) = \begin{vmatrix}3-\lambda & 4\\ 2 & 5-\lambda\end{vmatrix} = (3-\lambda)(5-\lambda)-8 = \lambda^2-8\lambda+7 = 0$$

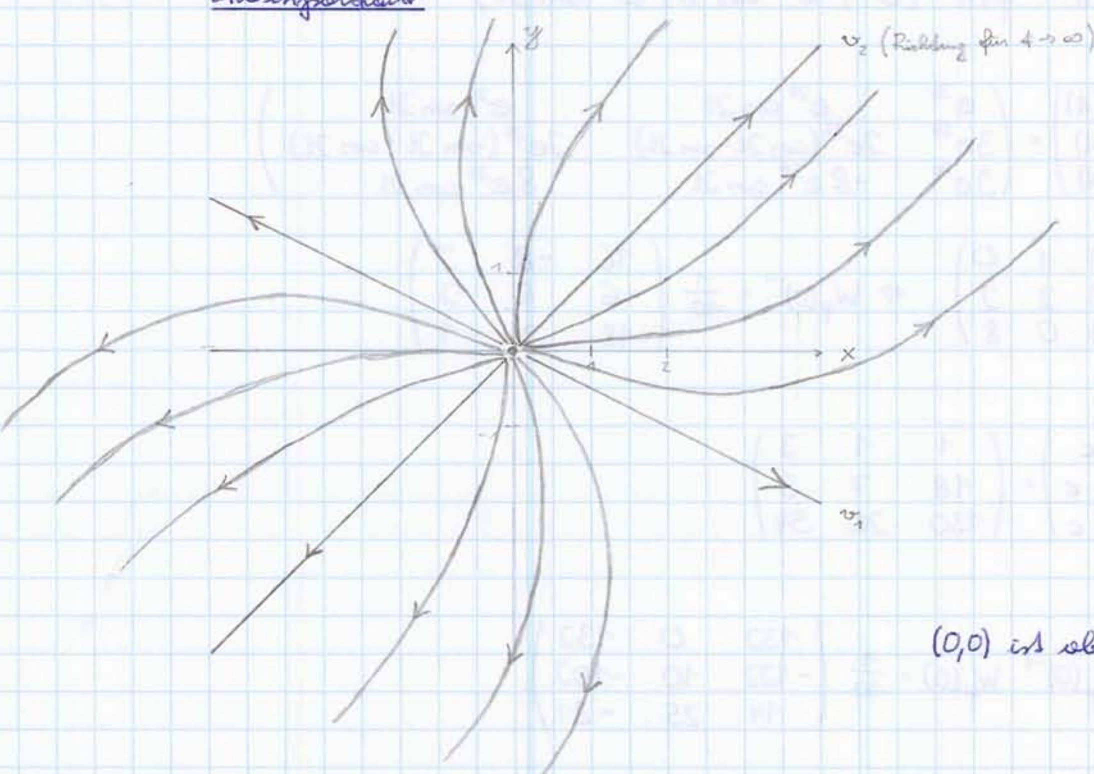
$$\Leftrightarrow \lambda = 4 \pm \sqrt{16-7} \Rightarrow \lambda_1 = 1,\ \lambda_2 = 7.$$

<u>char. Richtungen</u> $\hat{=}$ Eigenvektoren:

$\lambda_1 = 1$: $\quad A - E = \begin{pmatrix}2 & 4\\ 2 & 4\end{pmatrix} \Rightarrow v_1 = \begin{pmatrix}2\\ -1\end{pmatrix}$

$\lambda_2 = 7$: $\quad A - 7E = \begin{pmatrix}-4 & 4\\ 2 & -2\end{pmatrix} \Rightarrow v_2 = \begin{pmatrix}1\\ 1\end{pmatrix}$

<u>Phasenportrait:</u>



(0,0) ist abstoßender Knoten.

221) $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}}_{=:A} \begin{pmatrix} x \\ y \end{pmatrix}$

$$\chi_A(\lambda) = \begin{vmatrix} 1-\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - 2 = \lambda^2 - \lambda - 2 = 0$$

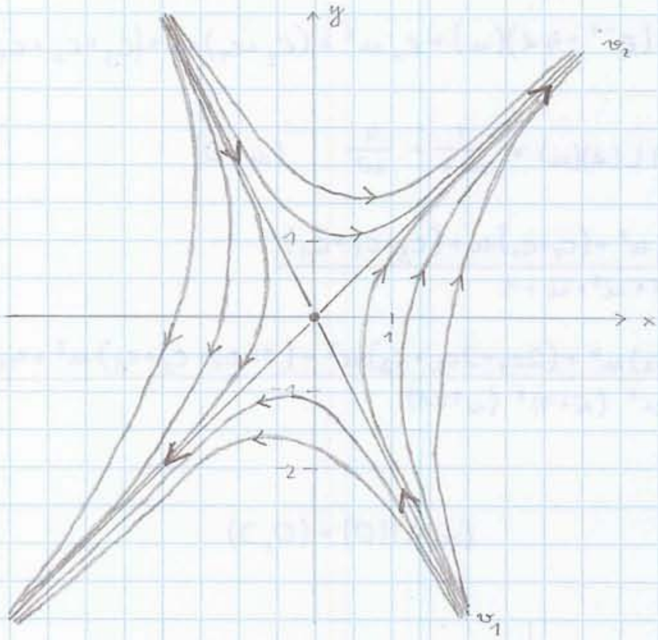
$$\Leftrightarrow \lambda = \frac{1}{2} \pm \sqrt{\frac{9}{4}} \Rightarrow \lambda_1 = -1,\ \lambda_2 = 2$$

<u>Char. Richtungen:</u>

$\lambda_1 = -1$: $\quad A + E = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda_2 = 2$: $\quad A - 2E = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

<u>Phasenportrait:</u>



$(0,0)$ ist Sattelpunkt.

230) $\displaystyle\lim_{y\to\infty} \int_0^1 x^y \, dx = \int_0^1 \lim_{y\to\infty} \underbrace{x^y}_{=: f(x,y)} dx$ wenn $\displaystyle\lim_{y\to\infty} x^y$ glm. konv. bzgl. $x \in [0,1]$

Es gilt $\forall$ $\displaystyle\underbrace{\lim_{y\to\infty} x^y}_{\varphi(x) :=} = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$

$$\Rightarrow |f(x,y) - \varphi(x)| = |x^y - \varphi(x)| = \begin{cases} x^y & x \in [0,1) \\ 0 & x = 1 \end{cases}$$

$\displaystyle\Rightarrow \sup_{x\in[0,1)} |f(x,y) - \varphi(x)| = \sup_{x\in[0,1)} x^y = 1 \quad \forall y \geq 0$ also nicht glm. konvergent.

<u>aber:</u> $\displaystyle\lim_{y\to\infty} \int_0^1 x^y \, dx = \lim_{n\to\infty} \int_0^1 x^n \, dx = \lim_{n\to\infty} \left(\frac{x^{n+1}}{n+1} \Big|_0^1 \right) = \lim_{n\to\infty} \frac{1}{n+1} = 0 = \int_0^1 \lim_{y\to\infty} x^y \, dx.$

239) $\begin{pmatrix} \dot x \\ \dot y \end{pmatrix} = \overbrace{\begin{pmatrix} 2 & -6 \\ 3 & -4 \end{pmatrix}}^{=:A} \begin{pmatrix} x \\ y \end{pmatrix} + \overset{b(t)}{\begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}}$ $\qquad \begin{pmatrix} x \\ y \end{pmatrix}(0) = \overset{c}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

$$(uE-A)\cdot L(\varphi)(u) = L(b)(u) + c$$

$$\Leftrightarrow \begin{pmatrix} u-2 & 6 \\ -3 & u+4 \end{pmatrix} \cdot L(\varphi)(u) = L\begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}(u) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{u-2}+1 \\ 0 \end{pmatrix} \qquad (u>2)$$

$$\Rightarrow L(\varphi)(u) = \begin{pmatrix} u-2 & 6 \\ -3 & u+4 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{u-2}+1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{u^2+2u+10} \begin{pmatrix} u+4 & -6 \\ 3 & u-2 \end{pmatrix} \begin{pmatrix} \frac{u-1}{u-2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{(u+4)(u-1)}{(u^2+2u+10)(u-2)} \\ \frac{3(u-1)}{(u^2+2u+10)(u-2)} \end{pmatrix}$$

243) $x^{(3)} + \ddot x + \dot x + x = e^{-t} + 4t$ $\qquad (x, \dot x, \ddot x)(0) = (c_1, c_2, c_3)$

$$(u^3+u^2+u+1)\, L(\varphi)(u) = L(e^{-t}+4t)(u) + c_1 u^2 + (c_2+c_1)u + (c_3+c_2+c_1)$$

$$L(e^{-t}+4t)(u) = L(e^{-t})(u) + 4L(t)(u) = \frac{1}{u+1} + \frac{4}{u^2} \qquad (u>0)$$

$$\Rightarrow L(\varphi)(u) = \frac{\frac{1}{u+1} + \frac{4}{u^2} + c_1 u^2 + (c_2+c_1)u + (c_3+c_2+c_1)}{u^3+u^2+u+1}$$

$$= \frac{c_1 u^5 + (2c_1+c_2)u^4 + (2c_1+2c_2+c_3)u^3 + (1+c_1+c_2+c_3)u^2 + 4u + 4}{u^2\,(u+1)^2\,(u^2+1)}$$

247) $\ddot x + (2t+1)\dot x + 3t x = t^3$ $\qquad (x, \dot x)(0) = (0, 2)$

$$\Leftrightarrow \ddot x + \dot x + 2t\dot x + 3tx = t^3$$

$$\Rightarrow \mathcal{L}(\ddot x)(u) + \mathcal{L}(\dot x)(u) + 2\mathcal{L}(t\dot x)(u) + 3\mathcal{L}(tx)(u) = \mathcal{L}(t^3)(u)$$

$$\mathcal{L}(\dot x)(u) = u\,\mathcal{L}(x)(u) - x(0) = u\,\mathcal{L}(x)(u)$$

$$\mathcal{L}(\ddot x)(u) = u\,\mathcal{L}(\dot x)(u) - \dot x(0) = u^2\mathcal{L}(x)(u) - 2$$

$$\mathcal{L}(tx)(u) = -\frac{d}{du}\mathcal{L}(x)(u)$$

$$\mathcal{L}(t\dot x)(u) = -\frac{d}{du}\mathcal{L}(\dot x)(u) = -\mathcal{L}(x)(u) - u\frac{d}{du}\mathcal{L}(x)(u)$$

$$\rightsquigarrow u^2\mathcal{L}(x)(u) - 2 + u\,\mathcal{L}(x)(u) - 2\mathcal{L}(x)(u) - 2u\frac{d}{du}\mathcal{L}(x)(u) - 3\frac{d}{du}\mathcal{L}(x)(u) = \frac{3!}{u^4}$$

$$\Leftrightarrow -(2u+3)\mathcal{L}'(x) + (u^2+u-2)\mathcal{L}(x) = \frac{6}{u^4} + 2.$$