21.4.10

Angewandte Statistik UE

III, 1) $X \sim L(\mu,\sigma^2)$ $(\Leftrightarrow \overbrace{\ln X}^{Y:=} \sim N(\mu,\sigma^2))$
$X_1,\dots,X_n$ Stichprobe

a)
Beobachtungsraum = Menge aller beobachtbaren Werte: $M_X = \mathbb{R}^+$

Stichprobenraum = Menge aller mögl. (n-dim.) Stichproben: $(\mathbb{R}^{+n}, \mathcal{B}_n^{\cap\mathbb{R}^+}, W^n)$ $W(B) := P[X\in B]\ \forall B\in \mathcal{B}\cap\mathbb{R}^+$

Parameterraum = Menge aller mögl. Parameter: $\Theta = \underset{\mu}{\mathbb{R}} \times \underset{\sigma^2}{\mathbb{R}^+}$

b) $Y \sim N(\mu,\sigma^2) \Rightarrow f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$

$y = \ln x \qquad \frac{\partial y}{\partial x} = \frac{1}{x} \qquad x \in \mathbb{R}^+$

$\Rightarrow X \sim L(\mu,\sigma^2)$ mit $f_X(x) = f_Y(\ln x)\cdot\frac{1}{x} = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2} \mathbb{1}_{(0,\infty)}(x)$

$\Rightarrow$ gemeinsame Dichte der Stichprobe:

$$f_{X_1,\dots,X_n}(x_1,\dots,x_n) = \prod_{k=1}^n f_X(x_k)$$
$$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \left(\prod_{k=1}^n x_k\right)^{-1} \exp\left[-\frac{1}{2\sigma^2}\sum_{k=1}^n (\ln x_k-\mu)^2\right] \mathbb{1}_{(0,\infty)^n}(x_1,\dots,x_n).$$

c) Stichprobenmittel zur Schätzung von μ:

$$\overline{Y}_n = \frac{1}{n}\sum_{i=1}^n \ln X_i = \frac{1}{n}\ln\left(\prod_{i=1}^n X_i\right)$$

Stichprobenvarianz zur Schätzung von σ^2:

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^n (\ln X_i - \overline{Y}_n)^2$$

2 a) Man drücke die ersten 4 zentralen Momente $\mu_k = \mathbb{E}[(X-\mathbb{E}X)^k]$ mit Hilfe der Momente um Null $\mu_k' = \mathbb{E}[X^k]$ aus.

$\mu_1 = \mathbb{E}[X-\mathbb{E}X] = 0$

$\mu_2 = \mathbb{E}[(X-\mathbb{E}X)^2] = \mathbb{E}[X^2] - 2(\mathbb{E}X)^2 + (\mathbb{E}X)^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \mu_2' - \mu_1'^2$

$\mu_3 = \mathbb{E}[(X-\mathbb{E}X)^3] = \mathbb{E}[X^3 - 3X^2\,\mathbb{E}X + 3X(\mathbb{E}X)^2 - (\mathbb{E}X)^3]$

$= \mathbb{E}X^3 - 3\mathbb{E}X\,\mathbb{E}X^2 + 3(\mathbb{E}X)^3 - (\mathbb{E}X)^3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3$

$\mu_4 = \mathbb{E}[(X-\mathbb{E}X)^4] = \mathbb{E}[X^4 - 4X^3\,\mathbb{E}X + 6X^2(\mathbb{E}X)^2 - 4X(\mathbb{E}X)^3 + (\mathbb{E}X)^4]$

$= \mathbb{E}X^4 - 4\mathbb{E}X^3\,\mathbb{E}X + 6\mathbb{E}X^2(\mathbb{E}X)^2 - 4(\mathbb{E}X)^4 + (\mathbb{E}X)^4$

$= \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

b) Die Beziehungen übertragen sich auf die empirischen Momente, da

$\mathbb{E} M_r = \mu_r$, $\mathbb{E} M_r' = \mu_r'$.

c) $\mu_r = \mathbb{E}[(X-\mathbb{E}X)^r] = \mathbb{E}\left[\sum_{k=0}^{r} \binom{r}{k} X^{r-k} (-\mathbb{E}X)^k\right]$

$$= \sum_{k=0}^{n} (-1)^k \, \mathbb{E}X^{r-k} (\mathbb{E}X)^k = \sum_{k=0}^{n} (-1)^k \mu_{r-k}' (\mu_1')^k$$

$$M_r = \sum_{k=0}^{n} (-1)^k M_{r-k}' (M_1')^k$$

3) $X_i \sim N(\mu, \sigma^2)$; ges.: $\mathrm{Var}\, S_n^2$

Berechne die Momente von X mithilfe der momenterzeugenden Fkt.:

$$m_X(s) = e^{\mu s + \frac{\sigma^2 s^2}{2}}$$

$$\left.\begin{aligned} \mu_1' &= m_X'(0) = \mu \\ \mu_2' &= m_X''(0) = \sigma^2 + \mu^2 \\ \mu_3' &= m_X^{(3)}(0) = 3\sigma^2\mu + \mu^3 \\ \mu_4' &= m_X^{(4)}(0) = 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4 \end{aligned}\right\} \Rightarrow \mu_4 = 3\sigma^4$$

1.4.2 b) $\rightsquigarrow$ $\mathrm{Var}\, S_n^2 = \frac{1}{n}\left(\mu_4 - \frac{n-3}{n-1}\sigma^4\right)$ für $n > 1$

$$= \frac{1}{n}\sigma^4 \underbrace{\left(3 - \frac{n-3}{n-1}\right)}_{\frac{3n-3-n+3}{n-1}} = \frac{2\sigma^4}{n-1}$$

4) ZZ: $\det A \geq 0$ für $A := \begin{pmatrix} 1 & \mu_1' & \mu_2' \\ \mu_1' & \mu_2' & \mu_3' \\ \mu_2' & \mu_3' & \mu_4' \end{pmatrix}$

$a^T \cdot A \cdot a \geq 0 \;\; \forall a \in \mathbb{R}^3 \Rightarrow \det A \geq 0$, da:

$\exists P \in \mathbb{R}^{3\times 3}$ mit $A = P^T \cdot B \cdot P$ wobei $B = \mathrm{diag}(E_{3-k}, \underbrace{0, \dots, 0}_{k\text{-mal}})$

1. Fall: $k \neq 0$: $\det(A) = \det(P^T \cdot B \cdot P) = \det(P^T) \cdot \underbrace{\det(B)}_{=0} \cdot \det(P) = 0$

2. Fall: $k = 0$: $\det(A) = \det(P^T) \cdot \underbrace{\det(B)}_{=1} \cdot \det(P) = (\det(P))^2 \geq 0$.

$$a^T \cdot A \cdot a = a_1^2 \cdot 1 + a_2^2 \cdot \mu_2' + a_3^2 \mu_4' + 2\mu_1' a_1 a_2 + 2\mu_2' a_1 a_3 + 2\mu_3' a_2 a_3$$

$$= a_1^2 + a_2^2 \, \mathbb{E}X^2 + a_3^2 \, \mathbb{E}X^4 + 2a_1 a_2 \, \mathbb{E}X + 2a_1 a_3 \, \mathbb{E}X^2 + 2a_2 a_3 \, \mathbb{E}X^3$$

$$= \mathbb{E}[a_1^2 + a_2^2 X^2 + a_3^2 X^4 + 2a_1 a_2 X + 2a_1 a_3 X^2 + 2a_2 a_3 X^3]$$

$$= \mathbb{E}[(a_1 + X a_2 + X^2 a_3)^2] \geq 0 \quad \forall a \in \mathbb{R}^3 \Rightarrow \det A \geq 0.$$

5) ges: Ausdruck in μ_i' für

a) $\mathrm{Cov}(M_r', M_s') = \mathbb{E}[(M_r' - \mathbb{E}M_r')(M_s' - \mathbb{E}M_s')]$

$$= \mathbb{E}[M_r' M_s'] - \underbrace{\mathbb{E}[M_r']\,\mathbb{E}[M_s']}_{\mu_r' \cdot \mu_s'}$$

$$\mathbb{E}[M_r' M_s'] = \mathbb{E}\left[\frac{1}{n^2}\left(\sum_{i=1}^{n} X_i^n\right)\left(\sum_{j=1}^{n} X_j^s\right)\right]$$

$$= \frac{1}{n^2}\,\mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} X_i^n X_j^s\right]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \underbrace{\mathbb{E}(X_i^n X_j^s)}_{\mathbb{E}X_i^n \cdot \mathbb{E}X_j^s \text{ da } X_i \perp X_j\ i\neq j} + \frac{1}{n^2}\sum_{i=1}^{n} \mathbb{E}X_i^{n+s}$$

$$= \frac{1}{n^2}\left(n(n-1)\,\mu_r'\mu_s'\right) + \frac{1}{n}\mu_{r+s}'$$

$$= \frac{n-1}{n}\mu_r'\mu_s' + \frac{1}{n}\mu_{r+s}'$$

$$\Rightarrow \mathrm{Cov}(M_r', M_s') = \frac{1}{n}\left(\mu_{r+s}' - \mu_r'\mu_s'\right)$$

b) $$\rho_{M_r', M_s'} = \frac{\mathrm{Cov}(M_r', M_s')}{\sqrt{\mathrm{Var}\,M_r'}\sqrt{\mathrm{Var}\,M_s'}} = \frac{\frac{1}{n}\left(\mu_{r+s}' - \mu_r'\mu_s'\right)}{\frac{1}{n}\sqrt{(\mu_{2r}' - \mu_r'^2)(\mu_{2s}' - \mu_s'^2)}}$$

$X \sim N(\mu, \sigma^2)$:

M_1', M_2' unkorreliert $\Leftrightarrow \mathrm{Cov}(M_1', M_2') = 0$

$\Leftrightarrow \mu_3' - \mu_1'\mu_2' = 3\sigma^2\mu + \mu^3 - \mu(\sigma^2 + \mu^2) = \mu(2\sigma^2) = 0$

$\Leftrightarrow \mu = 0$

$\Leftrightarrow X \sim N(0, \sigma^2)$

$$\rho_{M_1, M_2} = \frac{2\mu\sigma^2}{\sqrt{(\mu_2' - \mu_1')^2(\mu_4' - \mu_2')^2}} = \frac{2\mu\sigma^2}{\sqrt{2}\,\sigma^2\sqrt{\sigma^2 + 2\mu^2}} = \frac{\sqrt{2}\,\mu}{\sqrt{\sigma^2 + 2\mu^2}}$$

6) $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad \bar{x}_g = \sqrt[n]{\prod_{i=1}^{n} x_i} \qquad \bar{x}_h = \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}} \qquad x_i \geq 0 \;\; \forall i = 1,..,n$

Gleichheit für $x_i = x_j \;\; \forall i,j \in \{1,..,n\}$

ZZ: $\bar{x}_h \leq \bar{x}_g \leq \bar{x}$

$\varphi(x) := \ln x$ konkav $\quad (x \in \mathbb{R}^+)$

Jensen-Ungl. $\Rightarrow \mathbb{E}(\varphi(X)) \leq \varphi(\mathbb{E}(X))$

Sei $P(\{x_i\}) = \frac{1}{n}$, d.g.

$$\frac{1}{n}\sum_{i=1}^{n} \ln(x_i) \leq \ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \Leftrightarrow \ln\left(\left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}\right) \leq \ln\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)$$

$$\Leftrightarrow \bar{x}_g \leq \bar{x}$$

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \sqrt[n]{\prod_{i=1}^{n} x_i} \underset{\frac{1}{x_i} \geq 0}{\Rightarrow} \frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i} \geq \sqrt[n]{\prod_{i=1}^{n}\frac{1}{x_i}}$$

$$\Leftrightarrow \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}} \leq \sqrt[-n]{\prod_{i=1}^{n}\frac{1}{x_i}} = \sqrt[n]{\prod_{i=1}^{n} x_i} \Leftrightarrow \bar{x}_h \leq \bar{x}_g$$

$$m_r = \left[\sum_{i=1}^{n} \alpha_i x_i^r\right]^{1/r}, \quad \alpha_i \geq 0, \quad \sum_{i=1}^{n} \alpha_i = 1 \qquad (r \in \mathbb{R})$$

$\alpha_i := \frac{1}{n}$, $r = 1$: $\quad m_1 = \sum_{i=1}^{n} \frac{x_i}{n} = \bar{x}$

$r = -1$: $\quad m_{-1} = \frac{1}{\sum \frac{1}{n}\frac{1}{x_i}} = \frac{n}{\sum \frac{1}{x_i}} = \bar{x}_h$

$r = 0$: $\quad m_0 = \lim_{r \to 0}\left(\sum_{i=1}^{n} \frac{1}{n} x_i^r\right)^{\frac{1}{r}} = \sqrt[n]{\prod_{i=1}^{n} x_i}$

ZZ: $m(r)$ monoton $\nearrow$, d.h. $m(t) \geq m(s) \quad \forall t \geq s$

Def. $\varphi(x) = x^{\frac{t}{s}}$ konvex für $\frac{t}{s} > 1$.

Jensen-Ungl. $\Rightarrow \mathbb{E}(\varphi(X)) \geq \varphi(\mathbb{E}(X))$

Sei $P(\{x_i\}) = \alpha_i$, d.g.

$$\sum_{i=1}^{n} \alpha_i \varphi(x_i^s) = \sum_{i=1}^{n} \alpha_i x_i^t \geq \left(\sum_{i=1}^{n} \alpha_i x_i^s\right)^{\frac{t}{s}}$$

$$\Leftrightarrow \left(\sum_{i=1}^{n} \alpha_i x_i^t\right)^{\frac{1}{t}} \geq \left(\sum_{i=1}^{n} \alpha_i x_i^s\right)^{\frac{1}{s}}$$

7) $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ i.i.d.

$$V := \sum_{i=1}^n \alpha_i X_i \qquad W := \sum_{i=1}^n \beta_i X_i \qquad (\alpha_i, \beta_i \in \mathbb{R})$$

Wissen: $V, W \sim N(\cdot, \cdot)$

$$\mu_V = \mathbb{E}V = \mathbb{E} \sum_{i=1}^n \alpha_i X_i = \sum_{i=1}^n \alpha_i \mu$$

$$\sigma_V^2 = \mathrm{Var}\, V = \mathrm{Var} \sum_{i=1}^n \alpha_i X_i \underset{\perp}{=} \sum_{i=1}^n \alpha_i^2 \sigma^2 \qquad \Rightarrow V \sim N(\mu_V, \sigma_V^2)$$

$W \sim N(\mu_W, \sigma_W^2)$ analog

$$\mathrm{Cov}(V, W) = \mathbb{E}(V - \mathbb{E}V)(W - \mathbb{E}W)$$

$$= \mathbb{E}\left[\sum_{i=1}^n \alpha_i (X_i - \mu) \sum_{j=1}^n \beta_j (X_j - \mu) \right]$$

$$= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \beta_j \underbrace{\mathbb{E}(X_i - \mu)(X_j - \mu)}_{\substack{\mathrm{Cov}(X_i, X_j) \\ = \\ 0 \text{ für } i \neq j \\ \sigma^2 \quad i = j}} = \sum_{i=1}^n \alpha_i \beta_i \sigma^2$$

$$V \perp W \overset{\text{da NV}}{\Leftrightarrow} \mathrm{Cov}(V, W) = 0 \Leftrightarrow \alpha \perp \beta$$

gemeinsame Verteilung:

$$f_{V,W}(x) = \frac{1}{2\pi \sqrt{\det \Sigma}} \exp\left[-\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right]$$

$$= \frac{1}{2\pi \sqrt{\det \Sigma}} \exp\left[-\frac{1}{2} (x_1 - \mu_x, x_2 - \mu_y) \Sigma^{-1} \begin{pmatrix} x_1 - \mu_x \\ x_2 - \mu_y \end{pmatrix} \right]$$

mit $\Sigma = \begin{pmatrix} \sigma_V^2 & \mathrm{cov}(V, W) \\ \mathrm{cov}(V, W) & \sigma_W^2 \end{pmatrix}$