24.6.10

Analysis 3 UE

XII,

94. ges.: Fouriertransformierte von $f_1(x) := \frac{\sin x}{x}$, $f_2(x) := \frac{\cos x - 1}{x}$.

Achtung: $f_1, f_2 \notin L_1$ aber $\in L_2$

lt. Korollar 14.1.19: $\exists$ eindeutige, lineare, bijektive Abbildung $U: L_2(\mathbb{R}) \to L_2(\mathbb{R})$, die $\hat{\ }: L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \to L_2(\mathbb{R})$ fortsetzt. Es gilt $U(f) = \hat{f}$ für $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, $U \circ U(f) = f(-\cdot)$ für $f \in L_2(\mathbb{R})$.

a) Def.: $g(x) := \mathbb{1}_{[-1,1]}(x) \cdot \sqrt{\frac{\pi}{2}}$ (Bsp. 92)

$\Rightarrow g \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, $\hat{g}(x) = \frac{\sin x}{x} = f_1(x)$

$f_1 \in L_2 \Rightarrow U(f_1) = U(\hat{g}) = U \circ U(g) = g(-\cdot) = g$.

b) Def.: $h(x) := -\sqrt{\frac{\pi}{2}}\, i \cdot \operatorname{sgn}(x) \cdot \mathbb{1}_{[-1,1]}(x) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$

$\hat{h}(x) = \frac{\cos x - 1}{x} = f_2(x)$

$f_2 \in L_2 \Rightarrow U(f_2) = U(\hat{h}) = U \circ U(h) = h(-\cdot) = -h$.

Z34. ges.: Fouriertransformierte von $f(x) := \frac{1}{1+x^2}$.

Bezeichne $h(x) := e^{-x}\, \mathbb{1}_{[0,\infty)}(x)$ (Bsp. 93)

$$f(x) = \frac{1}{1+x^2} = \frac{1}{2}\left(\frac{1}{1+ix} + \frac{1}{1-ix}\right) = \frac{1}{2}\sqrt{2\pi}\left(\hat{h}(x) + \hat{h}(-x)\right)$$

$$\hat{h} \in L_1(\mathbb{R}) \Rightarrow \hat{f}(x) = \sqrt{\frac{\pi}{2}}\left(\hat{\hat{h}}(x) + \hat{\hat{h}}(-x)\right) = \sqrt{\frac{\pi}{2}}\left(h(-x) + h(x)\right)$$

$$= \sqrt{\frac{\pi}{2}}\left(e^{x}\, \mathbb{1}_{[0,\infty)}(-x) + e^{-x}\, \mathbb{1}_{[0,\infty)}(x)\right)$$

$\hat{\ } = \frac{1}{1+ix} \frac{1}{\sqrt{2\pi}}$ gilt noch immer!

$$= \sqrt{\frac{\pi}{2}}\, e^{-|x|} \cancel{+ \sqrt{\frac{\pi}{2}}\, \mathbb{1}_{\{0\}}(x)}.$$

~~Muss gelten:~~ $\cancel{\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, dt = \frac{1}{\sqrt{2\pi}} \arctan t \Big|_{-\infty}^{\infty} = \sqrt{\frac{\pi}{2}} = 2\sqrt{\frac{\pi}{2}} = \sqrt{2\pi}}$ ⚡

$$\Rightarrow \hat{f}(x) = \sqrt{\frac{\pi}{2}}\, e^{-|x|}.$$

Z40. ges: $\sigma(f)$ und $\mathcal{L}_c(f)(z)$ für $f_1(t) := t^n e^{wt}$, $f_2(t) := \sin wt$ $(w \in \mathbb{C}, n \in \mathbb{N})$.

a) Betrachte Funktion $f := t^n$, $n \in \mathbb{N}$.

$$\cdot)\ \mathcal{L}_c(f)(z) = \int_0^\infty t^n e^{-zt}\,dt = \underbrace{-\frac{1}{z} t^n e^{-zt}\Big|_{t=0}^{\infty}}_{0 \text{ für } \operatorname{Re} z > 0} + \int_0^\infty \frac{n}{z} t^{n-1} e^{-zt}\,dt$$

$$= \ldots = \frac{n!}{z^n} \int_0^\infty e^{-zt}\,dt = \frac{n!}{z^{n+1}} \left(-e^{-zt}\Big|_{t=0}^{\infty}\right) = \frac{n!}{z^{n+1}}$$

$$\cdot)\ \sigma(f) = \inf\left\{s \in \mathbb{R} : \int_{\mathbb{R}_0} |t^n \cdot e^{-st}|\,dt < \infty\right\} = 0 \quad \text{da } t^n e^{-st} \xrightarrow{t \to \infty} 0 \text{ für } s > 0.$$

Proposition 14.2.4 (6) $\rightsquigarrow$ $\mathcal{L}_c(f_1)(z) = \mathcal{L}_c(f)(z-w) = \dfrac{n!}{(z-w)^{n+1}}$

$\sigma(f_1) = \sigma(f) + \operatorname{Re} w = \operatorname{Re} w$ (muss also gelten: $\operatorname{Re} z > \operatorname{Re} w$)

b)

$$\cdot)\ \mathcal{L}_c(f_2)(z) = \int_0^\infty \sin wt\ e^{-zt}\,dt = \frac{1}{2i}\int_0^\infty (e^{iwt} - e^{-iwt})e^{-zt}\,dt$$

$$= \frac{1}{2i}\left(\int_0^\infty e^{-t(z-iw)}\,dt - \int_0^\infty e^{-t(z+iw)}\,dt\right)$$

$$\int |e^{-t(z\mp iw)}|\,dt = \int e^{-t\operatorname{Re}(z\mp iw)}\,dt = \int e^{-t(\operatorname{Re} z \mp \operatorname{Im} w)}\,dt < \infty \quad \text{für } \operatorname{Re} z \mp \operatorname{Im} w > 0 \Leftrightarrow \operatorname{Re} z > |\operatorname{Im} w|$$

$$= \frac{1}{2i}\left(\frac{1}{z-iw}\left(-e^{-t(z-iw)}\Big|_{t=0}^{\infty}\right) - \frac{1}{z+iw}\, e^{-t(z-iw)}\Big|_{t=0}^{\infty}\right)$$

$$= \frac{1}{2i}\left(\frac{1}{z-iw} - \frac{1}{z+iw}\right) = \frac{w}{z^2+w^2}$$

$\cdot)$ $\sigma(f_2) = |\operatorname{Im} w|$.

Z41. a) Sei $a > 0$, $g(x) := f(x-a)\, 1\!\!1_{[0,\infty)}(x-a)$.

<u>ZZ:</u> $\sigma(g) = \sigma(f)$, $\mathcal{L}_c(g)(z) = e^{-az}\mathcal{L}_c(f)(z)$.

$$\mathcal{L}_c(g)(z) = \int_{[0,\infty)} f(t-a)\, \underbrace{1\!\!1_{[0,\infty)}(t-a)}_{=1\!\!1_{[a,\infty)}(t)}\, e^{-zt}\, d\lambda(t)$$

$$= \int_{[a,\infty)} f(t-a)\, e^{-zt}\, d\lambda(t) = \int_{[0,\infty)} f(u)\, e^{-z(u+a)}\, d\lambda(u)$$

$$= e^{-az}\int_{[0,\infty)} f(u) e^{-zu}\, d\lambda(u) = e^{-az}\mathcal{L}_c(f)(z).$$

b) Sei $b \in \mathbb{C}$, $g(x) := f(x)\, e^{bx}$.

<u>ZZ:</u> $\sigma(g) = \sigma(f) + \operatorname{Re} b$, $\mathcal{L}_c(g)(z) = \mathcal{L}_c(f)(z-b)$.

$$\mathcal{L}_c(g)(z) = \int_{[0,\infty)} f(t)\, e^{bt}\, e^{-zt}\, d\lambda(t) = \int_{[0,\infty)} f(t)\, e^{-(z-b)t}\, d\lambda(t) = \mathcal{L}_c(f)(z-b)$$

$$\int |f(t) e^{bt} e^{-zt}|\, dt = \int |f(t)| \cdot e^{-(\operatorname{Re} z - \operatorname{Re} b)t}\, dt < \infty \quad \text{für } \operatorname{Re} z - \operatorname{Re} b > \sigma(f)$$

$$\Rightarrow \sigma(g) = \sigma(f) + \operatorname{Re} b.$$

c) Sei $f \in C^k[0,\infty)$, $\operatorname{Re} z > \max_{j=0\ldots k} \sigma(f^{(j)})$

<u>ZZ:</u> $\mathcal{L}_c(f^{(k)})(z) = z^k \mathcal{L}_c(f)(z) - \sum_{j=0}^{k-1} f^{(j)}(0)\, z^{k-1-j}$

Bew. mittels Induktion:

- $k=0$: $\mathcal{L}_c(f^{(0)})(z) = z^0 \mathcal{L}_c(f)(z)$ ✓
- $k \mapsto k+1$:

$$\mathcal{L}_c(f^{(k+1)}(z)) = \int_{[0,\infty)} f^{(k+1)}(t)\, e^{-zt}\, d\lambda(t)$$

$$= + f^{(k)}(t) e^{-zt}\Big|_{t=0}^{\infty} + z\int_{[0,\infty)} f^{(k)} e^{-zt}\, d\lambda(t)$$

$$= - f^{(k)}(0) + z\,\mathcal{L}_c(f^{(k)})(z)$$

$$= - f^{(k)}(0)\cdot z^0 + z^{k+1}\mathcal{L}_c(f)(z) - z\sum_{j=0}^{k-1} f^{(j)}(0)\, z^{k-1-j}$$

$$= z^{k+1}\mathcal{L}_c(f)(z) - \sum_{j=0}^{k} f^{(j)}(0)\, z^{k-j}.$$

Z42. Lösung von DGL mittels Laplace-Transformation.

a) $f''(t) + 4f'(t) = \cos 2t\,; \qquad f(0)=0,\; f'(0)=1$

$\cancel{\mathcal{L}(f''(t)) + 4\mathcal{L}(f')(z) = }$

$\mathcal{L}_c(f'')(z) + 4\mathcal{L}_c(f')(z) = z^2\mathcal{L}_c(f)(z) - f'(0) - f(0)\cdot z + 4(z\mathcal{L}_c(f)(z) - f(0)) = \mathcal{L}_c(\cos 2t)(z)$

$$\mathcal{L}_c(\cos 2t)(z) = \int_{[0,\infty)} \cos 2t\; e^{-zt}\, d\lambda(t) = \frac{1}{2}\int_{[0,\infty)} (e^{2it} + e^{-2it})\, e^{-zt}\, d\lambda(t)$$

$$= \frac{1}{2}\left(\int_{[0,\infty)} e^{-t(z-2i)}\, d\lambda(t) + \int_{[0,\infty)} e^{-t(z+2i)}\, d\lambda(t)\right)$$

$$= \frac{1}{2}\left(\frac{1}{2i-z}\, e^{-t(z-2i)}\Big|_{t=0}^{\infty} + \frac{1}{2i+z}\, e^{-t(z+2i)}\Big|_{t=0}^{\infty}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2i-z} + \frac{1}{2i+z}\right) = \frac{z}{z^2+4} \quad \text{für } \operatorname{Re} z > 0.$$

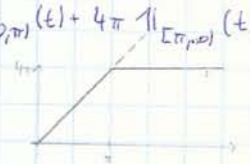
$$\Rightarrow \mathcal{L}_c(f)(z) = \frac{\frac{z}{z^2+4} + \overbrace{f'(0)}^{1} + \overbrace{f(0)\cdot z - 4f(0)}^{0}}{z^2+4z} = \frac{z^2+z+4}{z(z+4)(z^2+4)}$$

$$= \frac{1}{4}\,\frac{1}{z} - \frac{1}{20}\,\frac{z-4}{z^2+4} - \frac{1}{5}\,\frac{1}{z+4}$$

$$f(t) = \mathcal{L}_c^{-1}(\mathcal{L}_c(f)(z))(t) = \frac{1}{4}\mathcal{L}_c^{-1}\left(\frac{1}{z}\right)(t) - \frac{1}{20}\mathcal{L}_c^{-1}\left(\frac{z}{z^2+4}\right)(t) - \frac{1}{10}\mathcal{L}_c^{-1}\left(\frac{2}{z^2+4}\right)(t) - \frac{1}{5}\mathcal{L}_c^{-1}\left(\frac{1}{z+4}\right)(t)$$

$$= \frac{1}{4} - \frac{1}{20}\cos 2t - \frac{1}{10}\sin 2t - \frac{1}{5}e^{-4t}.$$

b) $f''(t) + 4f(t) = h(t)\,; \qquad f(0)=1,\; f'(0)=0, \quad h(t) := 4t\,\mathbb{1}_{[0,\pi)}(t) + 4\pi\,\mathbb{1}_{[\pi,\infty)}(t)$



$h(t) = 4t - (4t - 4\pi)\,\mathbb{1}_{[\pi,\infty)}(t) = 4t - 4(t-\pi)\,\mathbb{1}_{[0,\infty)}(t-\pi)$

$$\Rightarrow \mathcal{L}_c(h)(z) \overset{Z41}{=} \frac{4}{z^2} - 4e^{-\pi z}\frac{1}{z^2} = \frac{4}{z^2}(1 - e^{-\pi z})$$

$$\Rightarrow \mathcal{L}_c(f)(z) = \frac{\frac{4}{z^2}(1-e^{-\pi z}) + \overbrace{f(0)}^{1}\cdot z + \overbrace{f'(0)}^{0}}{z^2+4} = 4\,\frac{1-e^{-\pi z}}{z^2(z^2+4)} + \frac{z}{z^2+4} = \left(\frac{1}{z^2} - \frac{1}{z^2+4}\right)(1-e^{-\pi z}) + \frac{z}{z^2+4}$$

$$\Rightarrow f(t) = \mathcal{L}_c^{-1}(\mathcal{L}_c(f)(z))(t) = t - \frac{1}{2}\sin 2t - \left((t-\pi) - \frac{1}{2}\sin 2(t-\pi)\right)\mathbb{1}_{[0,\infty)}(t-\pi) + \cos 2t$$

$$= t - \frac{1}{2}\sin 2t - \left(t - \frac{1}{2}\sin 2t - \pi\right)\mathbb{1}_{[\pi,\infty)} + \cos 2t$$

$$= \begin{cases} t - \frac{1}{2}\sin 2t + \cos 2t & t \in [0,\pi) \\ \cos 2t + \pi & t \in [\pi,\infty) \end{cases}$$