17.6.10

Analysis 3 UE

$\underline{XI}$) 92.a) $f_1(x) := \mathbb{1}_{[-a,a]}(x)$

(i) <u>ges.:</u> $\hat{f}_1(x)$

$$\hat{f}_1(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-itx}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-itx}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \cos tx - i \sin tx\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{\sin tx}{x} \Big|_{t=-a}^{a} + i \underbrace{\frac{\cos tx}{x} \Big|_{t=-a}^{a}}_{0} \right) = \sqrt{\frac{2}{\pi}}\, \frac{\sin ax}{x}$$

(ii) $\hat{f}_1 \in L_1$?

$$\frac{1}{a} \int_{\mathbb{R}} |\hat{f}_1(x)|\, d\lambda(x) \overset{\sqrt{\frac{2}{\pi}}}{=} \int_{\mathbb{R}} \left| \frac{\sin ax}{ax} \right| d\lambda(x) = \sqrt{\frac{2}{\pi}}\, \frac{1}{a} \int_{\mathbb{R}} \left| \frac{\sin y}{y} \right| d\lambda(y) = \infty \quad \text{lt. Bsp. } 45$$

$\Rightarrow \hat{f}_1 \notin L_1$

(iii) $\hat{f}_1 \in L_2$?

$$\|f_1\|_2^2 = \int_{\mathbb{R}} \frac{\sin^2 ax}{x^2}\, d\lambda(x) = 2 \underbrace{\int_0^1 \frac{\sin^2 ax}{x^2}\, dx}_{\text{R-Int., da um 0 beschränkt}} + \underbrace{\int_1^\infty \frac{\sin^2 ax}{x^2}\, dx}_{\leq \int_1^\infty \frac{1}{x^2} dx = -\frac{1}{x}\big|_1^\infty = 1} < \infty .$$

b) $f_2(x) := \operatorname{sgn}(x)\, \mathbb{1}_{[-1,1]}(x)$

(i) <u>ges.:</u> $\hat{f}_2(x)$

$$\hat{f}_2(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-itx}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \operatorname{sgn}(t)\, e^{-itx}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-1}^{0} -e^{-itx}\, dt + \int_0^1 e^{-itx}\, dt \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(\underbrace{\frac{\sin tx}{x} \Big|_{t=0}^{1}}_{\frac{\sin x}{x}} + i \underbrace{\frac{\cos tx}{x} \Big|_{t=0}^{1}}_{\frac{\cos x - 1}{x}} - \underbrace{\frac{\sin tx}{x} \Big|_{t=-1}^{0}}_{-\frac{\sin(-x)}{x}} - i \underbrace{\frac{\cos tx}{x} \Big|_{t=-1}^{0}}_{+\frac{1-\cos(-x)}{x}} \right)$$

$$= \sqrt{\frac{2}{\pi}}\, i\, \frac{\cos x - 1}{x}$$

(ii) $\hat{f}_2 \in L_1$?

$$\int_{\mathbb{R}} \left|\frac{\cos x - 1}{x}\right| \, d\lambda(x) = 2\int_{\mathbb{R}^+} \frac{1-\cos x}{x} \, d\lambda(x) = 2\sum_{k=0}^{\infty} \int_{[k\pi, (k+2)\pi]} \frac{1-\cos x}{x} \, d\lambda(x)$$

$$\overset{1. \text{MWS}}{\geq} 2\sum_{k=0}^{\infty} \frac{1}{(k+2)\pi} \underbrace{\int_{k\pi}^{(k+2)\pi} 1-\cos x \, dx}_{2\pi} = 4 \sum_{k=0}^{\infty} \frac{1}{k+2} = \infty.$$

(iii) $\hat{f}_2 \in L_2$?

$$\int_{\mathbb{R}} \frac{(1-\cos x)^2}{x^2} \, d\lambda(x) = 2\int_{\mathbb{R}^+} \frac{(1-\cos x)^2}{x^2} \, d\lambda(x)$$

$$= 2\Big(\underbrace{\int_0^1 \frac{(1-\cos x)^2}{x^2} \, dx}_{\text{R-Int}, \lim_{x\to 0} \cdot = 0} + \underbrace{\int_1^{\infty} \frac{(1-\cos x)^2}{x^2} \, dx}_{\leq \int_1^{\infty} \frac{4}{x^2} dx = 4} \Big) < \infty.$$

93) a) $f(x) := e^{-x} \mathbb{1}_{[0,\infty)}(x)$

ges: $\hat{f}(x)$

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-t} e^{-itx} \, dt = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-t(1+ix)} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{-1}{1+ix} \underbrace{e^{-t(1+ix)} \Big|_{t=0}^{\infty}}_{-1} = \frac{1}{\sqrt{2\pi}} \frac{1}{1+ix}$$

b) Bestimme mittels Prop. 14.1.2 $\widehat{\operatorname{sgn}(x) e^{-|x|}}$.

$$\widehat{\operatorname{sgn}(x) e^{-|x|}} = \widehat{e^{-x} \mathbb{1}_{(0,\infty)}(x) - e^{x} \mathbb{1}_{(-\infty,0)}(x)} = \widehat{e^{-x} \mathbb{1}_{(0,\infty)}(x)} - \widehat{e^{x} \underbrace{\mathbb{1}_{(-\infty,0)}(x)}_{= \mathbb{1}_{(0,\infty)}(-x)}}$$

$$= \hat{f}(x) - \hat{f}(-x) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1+ix} - \frac{1}{1-ix} \right) = -\sqrt{\frac{2}{\pi}} \frac{ix}{1+x^2}$$

97) Sei f stetig; $f, \hat{f} \in L_1(\mathbb{R})$.

ges.: Funktionen $a(\lambda)$, $b(\lambda)$ sodass gilt:

$$u(x,0) = \int_0^\infty (a(\lambda)\cos(\lambda x) + b(\lambda)\sin(\lambda x))\,d\lambda = f(x) \quad \forall x \in \mathbb{R}.$$

Es gilt $\displaystyle\int_0^\infty a(\lambda)\cos\lambda x\,d\lambda = \frac{1}{2}\int_{-\infty}^\infty a(|\lambda|)\cos\lambda x\,d\lambda,$

$$\int_0^\infty b(\lambda)\sin\lambda x\,d\lambda = \frac{1}{2}\int_{-\infty}^\infty b(|\lambda|)\,\mathrm{sgn}(\lambda)\sin\lambda x\,d\lambda$$

$$\Rightarrow u(x,0) = \frac{1}{2}\int_{-\infty}^\infty a(|\lambda|)\cos\lambda x + b(|\lambda|)\,\mathrm{sgn}(\lambda)\sin\lambda x\,d\lambda$$

Unter Berücksichtigung von $e^{-i\lambda x} = \cos\lambda x - i\sin\lambda x$ und unter Annahme der Reellwertigkeit von $c(\lambda)$ lässt sich das folgendermaßen umschreiben:

$$u(x,0) = \int_{-\infty}^\infty \mathrm{Re}(c(\lambda)e^{-i\lambda x})\,d\lambda \quad \text{wobei: } c(\lambda) = a(|\lambda|) + i\,\mathrm{sgn}(\lambda)\,b(|\lambda|)$$

$$= \begin{cases} \frac{1}{2}(a(-\lambda) - i\,b(-\lambda)) & \lambda < 0 \\ \frac{1}{2}(a(\lambda) + i\,b(\lambda)) & \lambda > 0 \\ \frac{1}{2}a(0) & \lambda = 0 \end{cases}$$

$$\Rightarrow u(x,0) = \frac{1}{2}\int_{-\infty}^\infty \Big(c(\lambda)e^{-i\lambda x} + \underbrace{\overline{c(\lambda)e^{-i\lambda x}}}_{c(\lambda)e^{i\lambda x}}\Big)\,d\lambda$$

$$= \frac{1}{2}\left(\int_{-\infty}^\infty c(\lambda)e^{-i\lambda x}\,d\lambda + \int_{-\infty}^\infty c(\lambda)e^{i\lambda x}\,d\lambda\right) = \sqrt{\frac{\pi}{2}}\,\big(\hat{c}(x) + \underbrace{\hat{c}(-x)}_{=\hat{c}(-x) = \overline{\hat{c}(x)}}\big) \overset{\sqrt{2\pi}}{=} \mathrm{Re}(\hat{c}(x)) = f(x)$$

Gleichung erfüllt für $\overset{\sqrt{2\pi}}{\hat{c}}(x) = f(x) \iff c(-x) = \hat{\hat{c}}(x) = \hat{f}(x)\,\frac{1}{\sqrt{2\pi}}$

$$\iff c(x) = \hat{f}(-x)\,\frac{1}{\sqrt{2\pi}}$$

$a(\lambda) \overset{\frac{1}{\sqrt{2\pi}}}{:=} (\hat{f}(\lambda) + \hat{f}(-\lambda))$, $b(\lambda) \overset{\frac{1}{\sqrt{2\pi}}}{:=} i\,(\hat{f}(\lambda) - \hat{f}(-\lambda))$ leisten nun das Gewünschte.

98. a) <u>ZZ:</u> Funktion aus 97 lässt sich schreiben als $u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} f(z)\, e^{-\frac{(z-x)^2}{4t}}\, dz$.

Sei $t > 0$.

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \left([\hat{f}(\lambda) + \hat{f}(-\lambda)] \cos\lambda x + i [\hat{f}(\lambda) - \hat{f}(-\lambda)] \sin\lambda x \right) e^{-\lambda^2 t}\, d\lambda$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \hat{f}(\lambda) [\cos\lambda x + i \sin\lambda x]\, e^{-\lambda^2 t} + \hat{f}(-\lambda) [\cos\lambda x - i\sin\lambda x]\, e^{-\lambda^2 t}\, d\lambda$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_0^\infty \hat{f}(\lambda)\, e^{i\lambda x}\, e^{-\lambda^2 t}\, d\lambda + \int_0^\infty \hat{f}(-\lambda)\, e^{-i\lambda x}\, e^{-\lambda^2 t}\, d\lambda \right.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \hat{f}(\lambda)\, e^{i\lambda x}\, e^{-\lambda^2 t}\, d\lambda$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(z)\, e^{-iz\lambda}\, dz \right) e^{i\lambda x}\, e^{-\lambda^2 t}\, d\lambda$$

Fubini: $|f(u)\, e^{-i(z-x)\lambda - \lambda^2 t}| = \underbrace{|f(u)|}_{\in L_1(\mathbb{R}, \lambda(u))} \underbrace{|e^{-\lambda^2 t}|}_{\substack{\in \\ L_1(\mathbb{R}, \lambda(\lambda)) \\ (\int \dots = \sqrt{\frac{\pi}{t}})}} \in L_1(\mathbb{R}^2, \lambda(u) \times \lambda(\lambda))$

$$\overset{\text{Fubini}}{=} \frac{1}{2\pi} \int_{-\infty}^\infty f(z) \int_{-\infty}^\infty e^{-i(z-x)\lambda}\, e^{-\lambda^2 t}\, d\lambda\, dz$$

$$= \frac{1}{2\pi} \int_{-\infty}^\infty f(z) \int_{-\infty}^\infty [\cos(\lambda(z-x)) - i \sin(\lambda(z-x))]\, e^{-\lambda^2 t}\, d\lambda\, dz$$

$$\int_{-\infty}^\infty \cos(\lambda(z-x))\, e^{-\lambda^2 t}\, d\lambda = 2 \int_0^\infty \cos(\lambda(z-x))\, e^{-\lambda^2 t} \overset{\text{Hinweis}}{=} \sqrt{\frac{\pi}{t}}\, e^{-\frac{(z-x)^2}{4t}}$$

$$\int_{-\infty}^\infty \sin(\lambda(z-x))\, e^{-\lambda^2 t}\, d\lambda = \int_{\mathbb{R}} \sin(\lambda(z-x))\, e^{-\lambda^2 t} \lim_{N\to\infty} \mathbb{1}_{[-N,N]}$$

$$\overset{\text{Lebesgue}}{=} \lim_{N\to\infty} \underbrace{\int_{[-N,N]} \sin(\lambda(z-x))\, e^{-\lambda^2 t}}_{=0,\ \text{da ungerade Fkt.}} = 0$$

$$\Rightarrow u(x,t) = \frac{1}{2\pi} \int_{-\infty}^\infty f(z) \sqrt{\frac{\pi}{t}}\, e^{-\frac{(z-x)^2}{4t}}\, dz$$

$$= \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^\infty f(z)\, e^{-\frac{(z-x)^2}{4t}}\, dz = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} f(z)\, e^{-\frac{(z-x)^2}{4t}}\, dz.$$

b) <u>ZZ:</u> Die Funktion erfüllt die DGL $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$.

Für $t_0 > 0$ und $\delta \in (0, t_0)$ gilt

$$\left|\frac{\partial}{\partial t} f(z) e^{-\frac{(z-x)^2}{4t}}\right| = \left|f(z) e^{-\frac{(z-x)^2}{4t}} \frac{(z-x)^2}{4t^2}\right| \leq |f(z)| \frac{1}{(2t)^2} \left((z-x)^2 e^{-\frac{(z-x)^2}{4t}}\right)$$

$$\leq |f(z)| \underbrace{\frac{1}{(2(t_0-\delta))^2}}_{<\infty} (z-x)^2 e^{-\frac{(z-x)^2}{4(t_0+\delta)}} \quad \forall t \in (t_0-\delta, t_0+\delta)$$

Weiters gilt

$$\left\| f(z) \cdot (z-x)^2 e^{-\frac{(z-x)^2}{4(t_0+\delta)}} \right\|_1 \overset{\text{Hölder}}{\leq} \underbrace{\| f(z) \|_1}_{<\infty,\ \text{da } f \in L_1} \cdot \left\| (z-x)^2 e^{-\frac{(z-x)^2}{4(t_0+\delta)}} \right\|_\infty < \infty$$

$$\left\| (z-x)^2 e^{-\frac{(z-x)^2}{4(t_0+\delta)}} \right\|_\infty \cancel{= \| y^2 e^{-y^2} \|_\infty = e^{-1} < \infty}$$

$$= \left\| y^2 e^{-\frac{y^2}{4(t_0+\delta)}} \right\|_\infty = 4e^{-1}(t_0+\delta) < \infty$$

Es existiert also lokal eine L^1-Majorante, daher gilt lt. Lemma 13.1.10:

$$\frac{\partial u}{\partial t} = -\frac{1}{4\sqrt{\pi}\, t^{3/2}} \int_{\mathbb{R}} f(z) e^{-\frac{(z-x)^2}{4t}} dz + \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} f(z) e^{-\frac{(z-x)^2}{4t}} \frac{(z-x)^2}{4t^2} dz$$

Für $x_0 \in [0, \ell]$ gilt

$$\left|\frac{\partial}{\partial x} f(z) e^{-\frac{(z-x)^2}{4t}}\right| = \left|f(z) e^{-\frac{(z-x)^2}{4t}} \frac{z-x}{2t}\right| = |f(z)| e^{-\frac{(z-x)^2}{4t}} \frac{|z-x|}{2t}$$

$$\leq |f(z)| e^{-\frac{\min\{(z-(x_0-\delta))^2, (z-(x_0+\delta))^2\}}{4t}} \cdot \frac{\max\{|z-(x_0-\delta)|, |z-(x_0+\delta)|\}}{2t} \quad \forall x \in (x_0-\delta, x_0+\delta)$$

Wiederum gilt

$$\| \cdots \|_1 \leq \| f \|_1 \cdot \left\| \frac{z-x_1}{2t} e^{-\frac{(z-x_1)^2}{4t}} \right\|_\infty < \infty$$

und somit lt. Lemma:

$$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} f(z) e^{-\frac{(z-x)^2}{4t}} \frac{z-x}{2t} dz$$

Analog:

$$\left|\frac{\partial}{\partial x} f(z) e^{-\frac{(z-x)^2}{4t}} \frac{z-x}{2t}\right| = |f(z)| e^{-\frac{(z-x)^2}{4t}} \frac{1}{2t} \left|\frac{(z-x)^2}{2t} - 1\right|$$

$$\leq |f(z)| e^{-\frac{(z-x)^2}{4t}} \frac{1}{2t} \left(\frac{(z-x)^2}{2t} + 1\right) \leq |f(z)| e^{-\frac{\min \cdots}{4t}} \frac{1}{2t} \left(\frac{\max \cdots}{2t} + 1\right) \quad \forall x \in (x_0-\delta, x_0+\delta)$$

$$\| \cdots \|_1 \leq \| f \|_1 \cdot \left\| e^{-\frac{(z-x_1)^2}{4t}} \frac{1}{2t} \left(\frac{(z-x_1)^2}{2t} + 1\right) \right\|_\infty < \infty$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} f(z) e^{-\frac{(z-x)^2}{4t}} \frac{1}{2t} \left(\frac{(z-x)^2}{2t} - 1\right) dz$$

$$= \frac{1}{2\sqrt{\pi t}} \int_{\mathbb{R}} f(z) e^{-\frac{(z-x)^2}{4t}} \frac{(z-x)^2}{4t^2} dz - \frac{1}{4\sqrt{\pi}\, t^{3/2}} \int_{\mathbb{R}} f(z) e^{-\frac{(z-x)^2}{4t}} dz = \frac{\partial u}{\partial t}.$$

Z28. Sei $G := \{(x,y)^T \in \mathbb{R}^2 : x^2+y^2<1,\ y\neq 0\}$, $R = \binom{R_1}{R_2} : \overline{G} \to \mathbb{R}^2$ stetig, $R|_G \in C^1$.

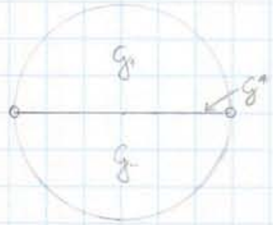
<u>ZZ:</u> $\int_G \operatorname{div} R \, d\lambda_2 = \int_{\partial G} \nu(u)^T R(u) \, d\mu_{\partial G}(u)$

Def.: $G_+ := \{(x,y)^T\in\mathbb{R}^2 : x^2+y^2<1,\ y>0\}$,

$G_- := \{$ ——— " ———, $y<0\}$,

$G^* := (-1,1)\times\{0\}$

•) G_+ offen, beschränkt



•) $R|_{\overline{G_+}} : \overline{G_+} \to \mathbb{R}^2$ stetig als Einschr.

$R|_{G_+}$ als Einschränkung C^1

•) supp $R|_{\overline{G_+}} \subseteq \overline{G_+} = G_+ \cup \partial^\circ G_+ \cup L$ mit $L=\{(-1,0)^T,(1,0)^T\}$ (⇒ Zusatzbed. erf.)

Gauß ⇒ $\int_{G_+} \operatorname{div} R \, d\lambda_2 = \int_{\partial^\circ G_+} \nu_{G_+}^T R \, d\mu_{\partial^\circ G_+}$

Analoges gilt für G_-.

$$\Rightarrow \int_G \operatorname{div} R\, d\lambda_2 = \int_{G_+} \operatorname{div} R\, d\lambda_2 + \int_{G_-} \operatorname{div} R\, d\lambda_2$$

$$= \int_{\partial G_+} \nu_{\partial G_+}^T R\, d\mu_{\partial G_+} + \int_{\partial G_-} \nu_{\partial G_-}^T R\, d\mu_{\partial G_-}$$

$$= \int_{\underbrace{(\partial G_+ \cup \partial G_-)\setminus G^*}_{=\partial G}} \nu^T R\, d\mu_{\partial G} + \int_{G^*} \big(\underbrace{\nu_{\partial G_+}^T + \nu_{\partial G_-}^T}_{(0,0,-1)+(0,0,1)=0}\big) R\, d\mu_{G^*} = \int_{\partial G} \nu^T R\, d\mu_{\partial G}.$$

Z33. μ endliches, positives Borelmaß auf $\mathbb{R}$, def. $\hat{\mu}(z) := \int_{\mathbb{R}} e^{-ixz}\, d\mu(x)$, $z \in \mathbb{R}$.

a) (i) <u>ZZ:</u> $\hat{\mu}$ beschränkt, $\|\hat{\mu}\|_\infty \leq \|\mu\| := \mu(\mathbb{R})$.

$$|\hat{\mu}(z)| = \left| \int_{\mathbb{R}} e^{-ixz}\, d\mu(x) \right| \leq \int_{\mathbb{R}} |e^{-ixz}|\, d\mu(x) = \mu(\mathbb{R}) < \infty \quad \forall z \in \mathbb{R}$$

$$\Rightarrow \|\hat{\mu}\|_\infty \leq \|\mu\|.$$

(ii) <u>ZZ:</u> $\hat{\mu}$ stetig

- $x \mapsto e^{-ixz}$ intbar
- $z \mapsto e^{-ixz}$ stetig
- 1 ist globale, von x unabh. $L^1(\mathbb{R}, \mu)$-Majorante

$\Rightarrow \hat{\mu}$ stetig.

b) <u>ges.:</u> Maß μ, sodass $\hat{\mu} \notin C_0(\mathbb{R})$

Def. $\mu(A) = \begin{cases} 1 & 0 \in A \\ 0 & \text{sonst} \end{cases}$

$$\Rightarrow \hat{\mu}(z) = e^{-ixz}\, 1\!\!1_{\{0\}}(x) = e^{-i0z} = 1 \quad \forall z \in \mathbb{R}$$

$$\Rightarrow \hat{\mu} \notin C_0(\mathbb{R})$$

c) <u>ZZ:</u> $t_1, \dots, t_n \in \mathbb{R}$, $n \in \mathbb{N} \Rightarrow$ Matrix $(\hat{\mu}(t_j - t_i))_{i,j=1}^n$ positiv semidefinit.

$A \in \mathbb{R}^{n \times n}$ ist pos. semidef., wenn $\bar{a}^T \cdot A \cdot a \geq 0 \quad \forall a \in \mathbb{C}^n$

$$\begin{aligned}
\bar{a}^T (\hat{\mu}(t_j - t_i))_{i,j=1}^n\, a &= \sum_{i=1}^n \sum_{j=1}^n \bar{a}_i\, \hat{\mu}(t_j - t_i)\, a_j \\
&= \sum_{i=1}^n \sum_{j=1}^n \int_{\mathbb{R}} \bar{a}_i\, a_j\, e^{-ix(t_j - t_i)}\, d\mu(x) \\
&= \int_{\mathbb{R}} \sum_{i=1}^n \sum_{j=1}^n \bar{a}_i\, e^{ixt_i}\, a_j\, e^{-ixt_j}\, d\mu(x) \\
&= \int_{\mathbb{R}} \left(\sum_{i=1}^n \overline{a_i e^{-ixt_i}} \right) \cdot \left(\sum_{j=1}^n a_j e^{-ixt_j} \right) d\mu(x) \\
&= \int_{\mathbb{R}} \underbrace{\left| \sum_{j=1}^n a_j\, e^{-ixt_j} \right|^2}_{\leq 0} d\mu(x) \geq 0.
\end{aligned}$$

Z30. Sei $p \in [1,\infty)$, $f \in L^p(\mathbb{R})$, $g(x) := \int_{(x,x+1)} f(t)\, d\lambda(t)$

<u>ZZ:</u> $g \in C_0(\mathbb{R}) = \{h \in C(\mathbb{R}) : \lim_{|x|\to\infty} h(x) = 0\}$

a) <u>ZZ:</u> $g \in C(\mathbb{R})$

$$|g(x)-g(y)| = \left| \int_{\mathbb{R}} f(t)\, 1\!\!1_{(x,x+1)}(t)\, d\lambda(t) - \int_{\mathbb{R}} f(t)\, 1\!\!1_{(y,y+1)}(t)\, d\lambda(t) \right|$$

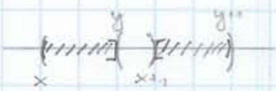
$$\leq \int_{\mathbb{R}} |f(t)|\, |1\!\!1_{(x,x+1)}(t) - 1\!\!1_{(y,y+1)}(t)|\, d\lambda(t)$$

$$= \int_{\mathbb{R}} |f(t)|\, 1\!\!1_{\underbrace{(x,x+1)\triangle(y,y+1)}_{=:A}}(t)\, d\lambda(t)$$

$$= \|f\, 1\!\!1_A\|_1 \leq \|f\|_p\, \|1\!\!1_A\|_{\frac{p}{p-1}} = \|f\|_p\, (\lambda(A))^{1-\frac{1}{p}}$$

Für $|x-y| < \delta < 1$ gilt:

$$\lambda(A) = \lambda((x,x+1)\triangle(y,y+1)) = \lambda((x\wedge y, x\vee y] \cup [x+1\wedge y+1, x+1 \vee y+1))$$



$$= 2(x\vee y - x\wedge y) < 2(x+\delta-(x-\delta)) = 4\delta$$

$$\Rightarrow |g(x)-g(y)| < \|f\|_p (4\delta)^{1-\frac{1}{p}} < \varepsilon \quad \forall y \in (x-\delta, x+\delta) \text{ mit } \delta < \frac{1}{4}\left(\frac{\varepsilon}{\|f\|_p}\right)^{\frac{p}{p-1}}.$$

$(\|f\|_p \neq 0)$

Für $\|f\|_p = 0$ trivial.

b) <u>ZZ:</u> $\lim_{x\to\pm\infty} g(x) = 0$

$$f \in L^p(\mathbb{R}),\ p \in [1,\infty) \Rightarrow \|f\|_p^p = \int_{\mathbb{R}} |f|^p\, d\lambda < \infty$$

$$\Rightarrow \lim_{x\to\infty} |f(x)|^p = 0 \Rightarrow \lim_{x\to\infty} |f(x)| = 0$$

$$\Rightarrow \lim_{x\to\infty} |g(x)| = \lim_{x\to\infty} \left| \int_{(x,x+1)} f(t)\, d\lambda(t) \right| \leq \lim_{x\to\infty} \int_{\mathbb{R}} |f(t)|\, 1\!\!1_{(x,x+1)}(t)\, d\lambda(t)$$

$$\leq \lim_{x\to\infty} \sup_{t\in(x,x+1)} |f(t)| \cdot \underbrace{\lambda((x,x+1))}_{1}$$

$$\leq \lim_{x\to\infty} \sup_{t>x} |f(t)| = \lim_{x\to\infty} \sup |f(x)| = \lim_{x\to\infty} |f(x)| = 0.$$

$\lim_{x\to-\infty} |g(x)| = 0$ analog.

Z31. Seien $p, q \in [1,\infty]$, $\frac{1}{p} + \frac{1}{q} = 1$; $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$.

a) <u>ZZ:</u> $t \to f(x-t)\, g(t)$ intbar $\forall x \in \mathbb{R}^d$, $\|f * g\|_\infty \leq \|f\|_p \cdot \|g\|_q$.

$$|f * g(x)| \leq \int_{\mathbb{R}^d} |f(x-t)\, g(t)|\, d\lambda_d(t) \overset{\text{Hölder}}{\leq} \left(\int_{\mathbb{R}^d} |f(x-t)|^p\, d\lambda_d(t)\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} |g(t)|^q\, d\lambda_d(t)\right)^{\frac{1}{q}}$$

$$= \|f\|_p \cdot \|g\|_q < \infty \quad \forall x \in \mathbb{R}^d$$

$$\Rightarrow \|f * g\|_\infty \leq \|f\|_p \cdot \|g\|_q$$

b) <u>ZZ:</u> $f * g$ glm. stetig

$$|f * g(x_1) - f * g(x_2)| \leq \int_{\mathbb{R}^d} |f(x_1 - t)\, g(t) - f(x_2 - t)\, g(t)|\, d\lambda_d(t)$$

$$= \int_{\mathbb{R}^d} |f(x_1 - t) - f(x_2 - t)|\, |g(t)|\, d\lambda_d(t)$$

$$\leq \left(\int_{\mathbb{R}^d} |f(x_1 - t) - f(x_2 - t)|^p\, d\lambda_d(t)\right)^{\frac{1}{p}} \cdot \|g\|_q$$

$$= \left(\int_{\mathbb{R}^d} |f(z - (x_2 - x_1)) - f(z)|^p\, d\lambda_d(t)\right)^{\frac{1}{p}} \cdot \|g\|_q$$

$$= \|f_{x_1 - x_2} - f\|_p\, \|g\|_q$$

f_t ist lt. Kor. 13.3.6 glm. stetig, d.h.

$$\forall \varepsilon > 0\ \exists \delta > 0: \|x_1 - x_2\| < \delta \Rightarrow \|f_{x_1 - x_2} - f\| < \frac{\varepsilon}{\|g\|_q}$$

$$\Rightarrow |f * g(x_1) - f * g(x_2)| < \varepsilon$$, also glm. stetig.