6.5.10

# Analysis 3 UE

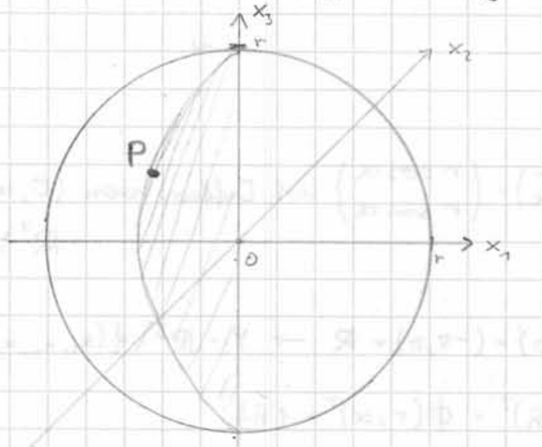
VII)

53. a) Kugelkoordinaten

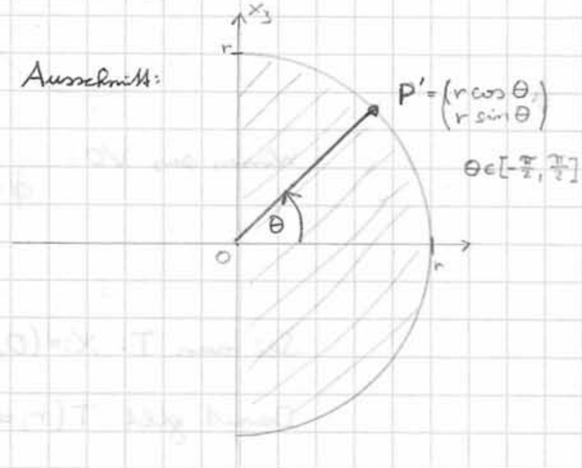
Sei $R := [0,\infty) \times [-\pi,\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, $T: R \to \mathbb{R}^3$ mit

$$T\begin{pmatrix} r \\ \alpha \\ \theta \end{pmatrix} := \begin{pmatrix} r\cos\alpha\cos\theta \\ r\sin\alpha\cos\theta \\ r\sin\theta \end{pmatrix}$$

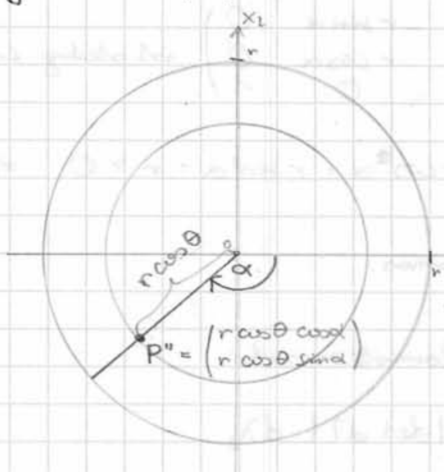
Motiviere die Bezeichnung aus der geom. Bedeutung der Parameter $r, \alpha, \theta$.




Ausschnitt:




Projektion in $x_1$-$x_2$-Ebene:



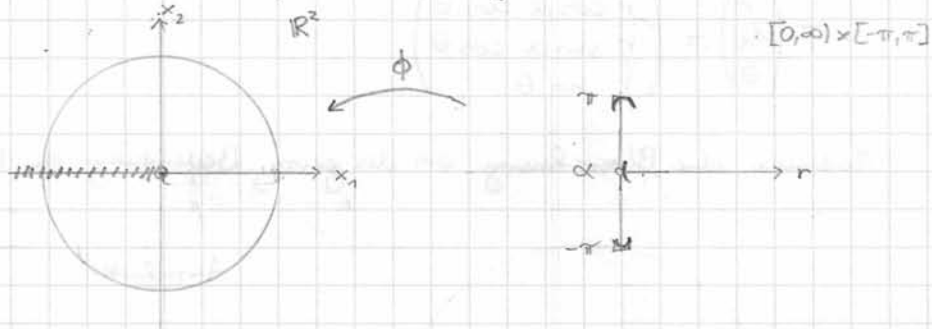

$$\Rightarrow P = \begin{pmatrix} r\cos\theta\cos\alpha \\ r\cos\theta\sin\alpha \\ r\sin\theta \end{pmatrix}$$

b) Zylinderkoordinaten

Sei $R := [0,\infty) \times [-\pi,\pi] \times \mathbb{R}$ und sei $T: R \to \mathbb{R}^3$ def. durch

$$T\begin{pmatrix} r \\ \alpha \\ h \end{pmatrix} := \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \\ h \end{pmatrix}$$

Betrachte zunächst Projektion des Zylinders in die $x_1$-$x_2$-Ebene:




Wissen aus VO:

$$\phi\begin{pmatrix} r \\ \alpha \end{pmatrix} = \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix} \text{ ist Diffeo. von } (0,\infty)\times(-\pi,\pi) \text{ auf } \mathbb{R}^2 \setminus \{x_1 \le 0 \wedge x_2 = 0\}$$

Sei nun $T: X := (0,\infty)\times(-\pi,\pi)\times\mathbb{R} \to Y := \mathbb{R}^3 \setminus \{(x_1,x_2,x_3)^T \in \mathbb{R}^3 : x_1 \le 0 \wedge x_2 = 0\}$.

Damit gilt $T(r,\alpha,h)^T = \phi(r,\alpha)^T \times \{h\}$.

$\Rightarrow$ •) $T$ ist offensichtlich bijektiv.

•) $$dT\begin{pmatrix} r \\ \alpha \\ h \end{pmatrix} = \begin{pmatrix} \cos\alpha & -r\sin\alpha & 0 \\ \sin\alpha & r\cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ ist stetig in allen Komp., also $T \in C^1$

•) $\det dT(r,\alpha,h)^T = r\cos^2\alpha + r\sin^2\alpha = r > 0 \Rightarrow T^{-1} \in C^1$

•) $X, Y$ offen in $\mathbb{R}^3$

$\Rightarrow$ $T$ ist $C^1$-Diffeomorphismus.

Daher folgt aus dem Transformationssatz:

$$\int_{\underbrace{T(X)}_{Y}} f \, d\lambda_3 = \int_X (f\circ T)\,|\det dT|\, d\lambda_3$$

$$\Rightarrow \int_{\mathbb{R}^3} f\, d\lambda_3 \underset{\uparrow}{=} \int_Y f\, d\lambda_3 = \int_X (f\circ T)\,|\det dT|\, d\lambda_3 \underset{\uparrow}{=} \int_R (f\circ T)\,|\det dT|\, d\lambda_3$$

$\mathbb{R}^3 \setminus Y$ ist Ebene $\Rightarrow \lambda_3$-NM

$\lambda_3(R\setminus X) = 0$, da
$R\setminus X = (\{0\}\times[-\pi,\pi]) \cup ([0,\infty)\times\{-\pi\}) \cup ([0,\infty)\times\{\pi\})$.

$B \in \mathcal{L}_2 \Rightarrow F(B) := \int_B 1 \, d\lambda_2$ heißt Fläche von $B$.

$B \in \mathcal{L}_3 \Rightarrow V(B) := \int_B 1 \, d\lambda_3$ heißt Volumen von $B$.

55. a) $B := \{(x,y,z)^T \in \mathbb{R}^3 : |x|+|y|+|z| < 1\}$ ; <u>ges.:</u> $V(B)$

$$(x_0, y_0, z_0)^T \in B \Leftrightarrow \begin{cases} x_0 \in \{x \in \mathbb{R} : |x| < 1\} =: A_1 \\ y_0 \in \{y \in \mathbb{R} : |y| < 1-|x_0|\} =: A_2 \\ z_0 \in \{z \in \mathbb{R} : |z| < 1-|x_0|-|y_0|\} =: A_3 \end{cases}$$

$$\Rightarrow \mathbb{1}_B(x,y,z) = \mathbb{1}_{A_1}(x) \cdot \mathbb{1}_{A_2}(y) \cdot \mathbb{1}_{A_3}(z)$$

$$\Rightarrow V(B) = \int_B 1 \, d\lambda_3 = \int \mathbb{1}_B(x,y,z) \, d\lambda_3(x,y,z) = \int \mathbb{1}_{A_1}(x) \cdot \mathbb{1}_{A_2}(y) \, \mathbb{1}_{A_3}(z) \, d\lambda_3(x,y,z)$$

$$\overset{\text{Fubini}}{=} \int_{A_1} \int_{A_2} \int_{A_3} 1 \, d\lambda(z) \, d\lambda(y) \, d\lambda(x)$$

$$= \int_{-1}^{1} \left( \int_{|x|-1}^{1-|x|} \underbrace{\left( \int_{|x|+|y|-1}^{1-|x|-|y|} 1 \, dz \right)}_{= 2(1-|x|-|y|)} dy \right) dx = 4 \cdot \int_{-1}^{1} \left( \int_{0}^{1-|x|} (1-|x|-y) \, dy \right) dx$$

$$= 4 \cdot \int_{-1}^{1} \left( 1-|x| - |x|(1-|x|) - \frac{(1-|x|)^2}{2} \right) dx = 8 \int_0^1 \left( \frac{1}{2} - x + \frac{x^2}{2} \right) dx = \frac{4}{3}.$$

b) $B' := \{(x,y,z)^T \in \mathbb{R}^3 : |a_{11}x + a_{12}y + a_{13}z| + |a_{21}x + a_{22}y + a_{23}z| + |a_{31}x + a_{32}y + a_{33}z| < 1\}$

<u>1. Fall:</u> $\det A \neq 0$

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} =: (\tilde{x}, \tilde{y}, \tilde{z})^T \quad ; \quad (x,y,z) \in B' \Leftrightarrow T(x,y,z) \in B$$

$$\Rightarrow \int_B 1 \, d\lambda_3(\tilde{x}, \tilde{y}, \tilde{z}) = \int_{B'} |\det A| \, d\lambda_3(x,y,z) \Leftrightarrow V(B') = V(B) \, |\det A|^{-1}.$$

<u>2. Fall:</u> $\det A = 0$

56. $a, b, c > 0$.

a) $B_1 := \{(x,y)^T \in \mathbb{R}^2 : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1\}$, ges.: $F(B)$

Koo.-Trafo:

$$\begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} r \\ \varphi \end{pmatrix} = \begin{pmatrix} a r \cos\varphi \\ b r \sin\varphi \end{pmatrix} \quad \Rightarrow \quad (x,y)^T \in B_1 \Leftrightarrow (r,\varphi)^T \in [0,1] \times [0, 2\pi]$$

$$F(B_1) = \int_{B_1} 1 \, d\lambda_2(x,y) \overset{\text{Bsp. 15.7.4}}{=} \int_{[0,1]\times[0,2\pi]} |\det dT| \, d\lambda_2(r,\varphi) \overset{\text{Fubini}}{=} a \cdot b \int_0^1 \Bigg( \underbrace{\int_0^{2\pi} 1 \, d\varphi}_{2\pi} \Bigg) \overset{r}{\cdot} dr = a b \pi$$

$$\det dT = \begin{vmatrix} a \cos\varphi & -a r \sin\varphi \\ b \sin\varphi & b r \cos\varphi \end{vmatrix} = a \cdot b \cdot r$$

b) $B_2 := \{(x,y,z)^T \in \mathbb{R}^3 : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1\}$, ges.: $V(B_2)$

Koo.-Trafo:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = T\begin{pmatrix} r \\ \alpha \\ \theta \end{pmatrix} = \begin{pmatrix} a r \cos\alpha \cos\theta \\ b r \sin\alpha \cos\theta \\ c r \sin\theta \end{pmatrix} \quad \Rightarrow \quad (x,y,z)^T \in B_2 \Leftrightarrow (r,\alpha,\theta)^T \in \underbrace{[0,1] \times [-\pi,\pi] \times [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]}_{=: R'}$$

$$V(B_2) = \int_{B_2} 1 \, d\lambda_3 = \int_{\mathbb{R}^3} 1\!\!1_{B_2}(x,y,z) \, d\lambda_3(x,y,z) \overset{\text{Bsp. 52}}{=} \int_R 1\!\!1_{R'}(r,\alpha,\theta) \overset{|\det dT|}{\cdot} d\lambda_3(r,\alpha,\theta)$$

$$= \int_{R'} |\det dT| \, d\lambda_3(r,\alpha,\theta) = \int_{R'} r^2 \cos\theta \, d\lambda_3(r,\alpha,\theta) \cdot abc$$

$$\overset{\substack{\text{Fubini,} \\ (\cos\theta \geq 0)}}{=} \int_{[0,1]} \int_{[-\pi,\pi]} \int_{[-\frac{\pi}{2},\frac{\pi}{2}]} r^2 \cos\theta \, d\lambda(\theta) \, d\lambda(\alpha) \, d\lambda(r) \cdot abc$$

$$= \int_0^1 \int_{-\pi}^{\pi} \underbrace{\left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 \cos\theta \, d\theta \right)}_{2r^2 \sin\theta \big|_{\theta=0}^{\frac{\pi}{2}} = 2r^2} d\alpha \, dr \overset{\cdot abc}{\cdot} = 4 \int_0^1 r^2 \Bigg( \underbrace{\int_0^{\pi} d\alpha}_{\pi} \Bigg) dr = \frac{4}{3}\pi \cdot a \cdot b \cdot c$$

~~58.~~

57. $a, b \in \mathbb{R}$ ; $a < b$ ; $f : [a,b] \to [0, \infty)$

$F := \{(x,y)^T \in \mathbb{R}^2 : 0 \le y \le f(x),\ a \le x \le b\}$ rotiert um x Achse und erzeugt Drehkörper $B_F$.

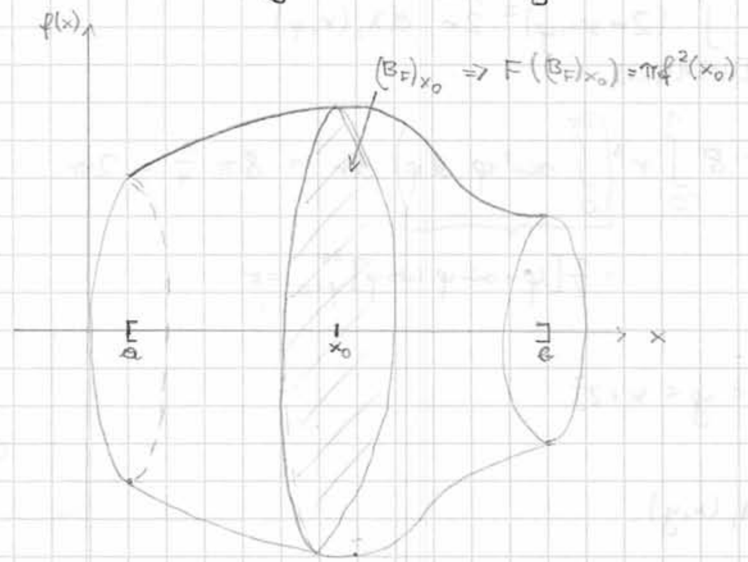
a) <u>ZZ:</u> $V(B_F) = \pi \int_{[a,b]} f^2 \, d\lambda$.

$B_F = \{(x,y,z)^T \in \mathbb{R}^3 : x \in [a,b] \wedge \sqrt{y^2+z^2} \le f(x)\}$

$$\Rightarrow V(B_F) = \int_{B_F} 1 \, d\lambda_3^{(x,y,z)} \overset{\text{Fubini}}{=} \int_{[a,b]} \left( \int_{(B_F)_x} 1 \, d\lambda(y,z) \right) d\lambda(x) \quad \text{mit } (B_F)_x = \{(y,z) \in \mathbb{R}^2 : \sqrt{y^2+z^2} \le f(x)\}$$

$$= \int_{[a,b]} \lambda_2\big(\underbrace{(B_F)_x}_{K_{f(x)}(x)}\big) \, d\lambda(x) = \int_{[a,b]} f^2(x)\pi \, d\lambda(x) = \pi \int_{[a,b]} f^2 \, d\lambda.$$

b) Motivation aus geom. Anschauung:




59. a) $$I_1 := \int_{[a,b]\times[c,d]} xy\, e^{x^2+y^2} \, d\lambda_2(x,y) \overset{\text{Fubini (f. intbar)}}{=} \int_{[a,b]} \left( \int_{[c,d]} x e^{x^2}\, y e^{y^2} \, d\lambda(y) \right) d\lambda(x)$$

$$= \int_a^b x e^{x^2} \, dx \cdot \int_c^d y e^{y^2} \, dy = \frac{1}{2} \int_{a^2}^{b^2} e^u \, du \cdot \frac{1}{2} \int_{c^2}^{d^2} e^v \, dv$$

$$= \frac{1}{4} \left(e^{b^2} - e^{a^2}\right)\left(e^{d^2} - e^{c^2}\right).$$

b) $B := \{(x,y) : 4x^2+y^2 < 4\} \Rightarrow B_x = \{y : y^2 < 4-4x^2\} = (-2\sqrt{1-x^2}, 2\sqrt{1-x^2})$

$$\Rightarrow I_2 := \int_B y^2 \, d\lambda_2(x,y) = \int_{(-1,1)} \int_{B_x} y^2 \, d\lambda(y) \, d\lambda(x)$$

$$= \int_{-1}^{1} \left( \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} y^2 \, dy \right) dx = \dots = \frac{2^4}{3} \int_{-1}^{1} (1-x^2)^{\frac{3}{2}} \, dx$$

$$= \frac{2^4}{3} \left[ \frac{1}{4} x (1-x^2)^{\frac{3}{2}} + \frac{3}{8} \sqrt{1-x^2} + \frac{3}{8} \arcsin x \right]_{x=-1}^{x}$$

$$= \frac{2^4}{3} \frac{3}{8} \pi = 2\pi$$

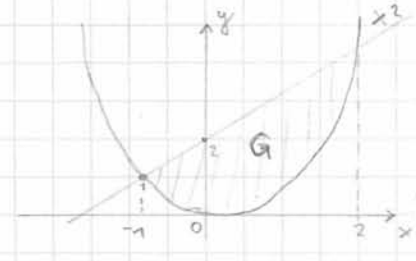
<u>Alternative:</u>

$$B = \{(x,y) \in \mathbb{R}^2 : x^2 + \left(\tfrac{y}{2}\right)^2 < 1\}$$

analog Bsp. 56 a), mit $a := 1$, $b := 2$:

$$\int_B y^2 \, d\lambda_2(x,y) = \int_{[0,1]\times[0,2\pi]} (2r \sin\varphi)^2 \overbrace{2r}^{\det T} \, d\lambda_2(r,\varphi)$$

$$= 8 \int_0^1 r^3 \underbrace{\left( \int_0^{2\pi} \sin^2\varphi \, d\varphi \right)}_{= \frac{1}{2}[\varphi - \sin\varphi\cos\varphi]_{\varphi=0}^{2\pi} = \pi} dr = 8\pi \frac{1}{4} = 2\pi.$$




c) $G := \{(x,y) \in \mathbb{R}^2 : x^2 < y < x+2\}$

$$I_3 := \int_G xy \, d\lambda_2(x,y)$$

$$= \int_{(-1,2)} \left( \int_{G_x} xy \, d\lambda(y) \right) d\lambda(x)$$

$$= \int_{-1}^{2} x \underbrace{\left( \int_{x^2}^{x+2} y \, dy \right)}_{\frac{1}{2}[(x+2)^2 - x^4]} dx = \frac{1}{2} \int_{-1}^{2} x^3 + 4x^2 + 4x - x^5 \, dx = \frac{45}{8}.$$

62) $A \in \mathbb{R}^{n\times n}$ pos. definit (somit ~~Abb~~ Koord.-Matrix einer symm. BLF)

quadratische Form $Q(x) = x^T A x \quad (x \in \mathbb{R}^n)$

orthogon. Gruppe

A symmetrisch $\Rightarrow \exists P \in O(n)$ sodass $P \cdot A \cdot P^T = D = \mathrm{diag}(\ell_1, \dots, \ell_n)$ mit $\ell_i$ = EW von $A$.

A pos. def. $\Rightarrow \ell_i > 0 \quad \forall i$ $\qquad\qquad \Updownarrow$

$$A = P^T \cdot D \cdot P$$

$$\Rightarrow Q(x) = x^T A x = x^T P^T D P x = \underbrace{(Px)}_{=:y}{}^T \cdot D \cdot (Px) = y^T \cdot D \cdot y = \sum_{i=1}^n \ell_i y_i^2$$

Trafosatz:

$$\Rightarrow \int_{\mathbb{R}^n} e^{-Q(x)} \, d\lambda_n(x) = \int_{\mathbb{R}^n} e^{-\sum \ell_i y_i^2} \underbrace{|\det P|}_{=1} \, d\lambda_n(y)$$

Fubini

$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} e^{-\sum \ell_i y_i^2} \, d\lambda(y_1) \dots d\lambda(y_n)$$

$$= \prod_{i=1}^n \int_{\mathbb{R}} e^{-\ell_i y_i^2} \, d\lambda(y_i)$$

$$= \prod_{i=1}^n \sqrt{\frac{\pi}{\ell_i}} = \pi^{\frac{n}{2}} \cdot \prod_{i=1}^n \ell_i^{-\frac{1}{2}} .$$