8.6.09

Analysis UE

X1, 221, 235, 238, 243, 248, 250, 255, 260

221)
$$f(x) = \sum_{n=1}^{\infty} \underbrace{\frac{(-1)^n}{n^2}}_{=: a_n} \cos nx \qquad [-\pi, \pi]$$

f wird durch obige Reihe beschrieben, reine Cosinusreihe $\Rightarrow$ f gerade $\Rightarrow$ f' unger.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx$$

$$= \frac{1}{\pi} \left(f(x) \frac{\sin nx}{n} \Big|_{-\pi}^{\pi} - \frac{1}{n} \int_{-\pi}^{\pi} f'(x) \sin nx \, dx \right.$$

$$= \frac{1}{\pi n} \Big(\underbrace{-\frac{1}{n} f'(x) \cos nx \Big|_{-\pi}^{\pi}} + \frac{1}{n} \int_{-\pi}^{\pi} f''(x) \cos nx \, dx \Big)$$

$$= -\frac{1}{n} \left(f'(\pi) \cos n\pi - f'(-\pi) \cos n\pi \right)$$

$$\overset{f' \text{ unger.}}{=} -\frac{2}{n} f'(\pi) \cos n\pi = -\frac{2}{n} f'(\pi) (-1)^n$$

$$\Rightarrow a_n = \frac{(-1)^n}{n^2} = \frac{(-1)^n}{n^2} \underbrace{\frac{2}{\pi} f'(\pi)}_{\text{soll 1 sein}} - \underbrace{\frac{1}{\pi n^2} \int_{-\pi}^{\pi} f''(x) \cos nx \, dx}_{\text{soll 0 sein}}$$

soll 0 sein $\Rightarrow$ f'' ist Konstante $\Rightarrow$ $f'(x)$ ist Gerade (*)

$$\Updownarrow$$

$$f'(\pi) = \frac{\pi}{2}$$

$$(*) \Rightarrow f'(x) = \frac{x}{2} \Rightarrow f(x) = \frac{x^2}{4} + c$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos 0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{x^2}{4} + c \right) dx$$

$$= \frac{1}{\pi} \left(\frac{2\pi^3}{12} + 2\pi c \right) = \frac{\pi^2}{6} + 2c$$

$$a_0 = 0 \Leftrightarrow c = -\frac{\pi^2}{12}$$

$$\Rightarrow f(x) \overset{\text{Dirichlet}}{=} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \overset{\text{Dirichlet}}{=} \underline{\frac{x^2}{4} - \frac{\pi^2}{12}.}$$

235) $(\mathbb{R}^k, \|\cdot\|_\Sigma)$, $(\mathbb{R}^m, \|\cdot\|_{max})$ ges.: induzierte Matrizennorm auf M_{mk}

$$\|(a_{ij})\| = \sup_{\|x\|_\Sigma = 1} \max_{1\le j\le m} \left| \sum_{i=1}^{k} a_{ji} \cdot x_i \right|$$

$$\le \sup_{\|x\|_\Sigma = 1} \max_{1\le j\le m} \sum_{i=1}^{k} |a_{ji}| \cdot |x_i|$$

$$\le \sup_{\|x\|_\Sigma = 1} \max_{1\le j\le m} \sum_{i=1}^{k} \max_{1\le i\le k} (|a_{ji}|) \cdot |x_i|$$

$$= \sup_{\|x\|_\Sigma = 1} \max_{1\le j\le m} \max_{1\le i\le k} (|a_{ji}|) \cdot \sum_{i=1}^{k} |x_i|$$

$$= \max_{1\le j\le m} \max_{1\le i\le k} |a_{ji}|$$

$$\Rightarrow \left\| \begin{pmatrix} 3 & 2 & -6 \\ 1 & 5 & -1 \end{pmatrix} \right\| = 6$$

238) Richtungsableitungen, Diffbarkeit in $(0,0)$.

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & (x,y) \ne (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$f'_{(h,k)}(0,0) = \lim_{\lambda\to 0} \frac{f(0+\lambda h, 0+\lambda k) - f(0,0)}{\lambda}$$
$$= \lim_{\lambda\to 0} \frac{\cancel{\lambda^3} h^3}{\cancel{\lambda^3}(h^2+k^2)} = \frac{h^3}{h^2+k^2}$$

also: alle Richtungsabl. in $(0,0)$ $\exists \Rightarrow f$ in $(0,0)$ Gâteaux-diffbar.

$f'_{(h,k)}$ keine lin. Funktion $\Rightarrow f$ kann in $(0,0)$ nicht (Fréchet-) diffbar sein.

250) Funktionalmatrix; Beschränktheit, ~~Stetigkeit~~ Stetigkeit der Matrix in Umg. v. $(0,0)$

$$\frac{\partial f}{\partial x}(x,y) = \frac{3x^2(x^2+y^2) - 2x^4}{(x^2+y^2)^2} = \frac{x^4 + 3x^2y^2}{(x^2+y^2)^2} = \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2} \quad \forall (x,y) \ne (0,0)$$

$$\frac{\partial f}{\partial x}(0,0) = f'_{(1,0)}(0,0) = \lim_{\lambda\to 0} \frac{f(0+\lambda,0) - f(0,0)}{\lambda} = \lim_{\lambda\to 0} \frac{\frac{\lambda^3}{\lambda^2}}{\lambda} = 1$$

$$\frac{\partial f}{\partial y}(x,y) = -\frac{x^3 \cdot 2y}{(x^2+y^2)^2} \quad \forall (x,y) \ne (0,0)$$

$$\frac{\partial f}{\partial y}(0,0) = f'_{(0,1)}(0,0) = \lim_{\lambda\to 0} \frac{f(0,0+\lambda) - f(0,0)}{\lambda} = \lim_{\lambda\to 0} \frac{\frac{0}{\lambda^2}}{\lambda} = 0$$

$$\Rightarrow \frac{df}{d(x,y)}(x,y) = \left(\frac{x^2(x^2+3y^2)}{(x^2+y^2)^2}, -\frac{2x^3 y}{(x^2+y^2)^2} \right) \quad \forall (x,y) \ne (0,0)$$

$\frac{df}{}(0,0) \; (\; ,\;)$

Beschränktheit von $\frac{df}{d(x,y)}$ in Umg. von $(0,0)$:

$$\frac{\partial f}{\partial x}(x,y) = \frac{x^4+3x^2y^2}{(x^2+y^2)^2} \leq \frac{4\,\|(x,y)\|_{max}^4}{\|(x,y)\|_{max}^4} = 4 \qquad \forall (x,y) \neq (0,0)$$

$$\frac{\partial f}{\partial y}(x,y) = -\frac{2x^3y}{(x^2+y^2)^2} \leq \left|\frac{2x^3y}{(x^2+y^2)^2}\right| \leq \frac{2\,\|(x,y)\|_{max}^4}{\|(x,y)\|_{max}^4} = 2 \qquad \text{—"—}$$

also part. Ableitungen beschr. $\Rightarrow$ f in $(0,0)$ stetig.

Stetigkeit von $\frac{df}{d(x,y)}$ in $(0,0)$:

•) $\frac{\partial f}{\partial x}(x,y)$ in $(0,0)$ stetig $\Leftrightarrow \lim\limits_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y) = 1$

Annäherung auf der y-Achse:

$$\Rightarrow \lim_{y\to 0} \frac{\partial f}{\partial x}(0,y) = \frac{0}{y^4} = 0$$

•) $\frac{\partial f}{\partial y}(x,y)$ in $(0,0)$ stetig $\Leftrightarrow \lim\limits_{(x,y)\to(0,0)} \frac{\partial f}{\partial y}(x,y) = 0$

Annäherung: $y=x$

$$\Rightarrow \lim_{x\to 0} \frac{\partial f}{\partial y}(x,x) = -\frac{2x^4}{(2x^2)^2} = -\frac{1}{2}$$

also part. Abl. in $(0,0)$ unstetig $\Rightarrow$ f in $(0,0)$ nicht diffbar

243)

$$f(x,y) = \begin{cases} \frac{xy}{e^{xy}-1} & xy \neq 0 \\ 1 & xy = 0 \end{cases}$$

$$f'_{(h,k)}(0,0) = \lim_{\lambda\to 0} \frac{f(\lambda h, \lambda k) - f(0,0)}{\lambda}$$

$$= \lim_{\lambda\to 0} \frac{\lambda^2 hk - e^{\lambda^2 hk} + 1}{\lambda\,(e^{\lambda^2 hk} - 1)} = „\frac{0}{0}“$$

$$= \lim_{\lambda\to 0} \frac{2\lambda hk\,(1-e^{\lambda^2 hk})}{e^{\lambda^2 hk}(2\lambda^2 hk + 1) - 1} = „\frac{0}{0}“$$

$$= \lim_{\lambda\to 0} \frac{2\cancel{hk}(1-e^{\lambda^2 hk}) + \cancel{2}\lambda \cancel{hk}(-2\lambda hk\, e^{\lambda^2 hk})}{\cancel{2}\lambda \cancel{hk}\, e^{\lambda^2 hk}(2\lambda^2 hk + 1) + e^{\lambda^2 hk}\cdot \cancel{4}_{2}\lambda \cancel{hk}}$$

$$= \lim_{\lambda\to 0} \frac{1 - e^{\lambda^2 hk}(1-2\lambda^2 hk)}{e^{\lambda^2 hk}(2\lambda^3 hk + 3\lambda)} = „\frac{0}{0}“$$

$$= \lim_{\lambda\to 0} \frac{-2\lambda hk\,\cancel{e^{\lambda^2 hk}}(1-2\lambda hk) + \cancel{e^{\lambda^2 hk}}\cdot 4\lambda hk}{2\lambda hk\,\cancel{e^{\lambda^2 hk}}(2\lambda^3 hk + 3\lambda) + \cancel{e^{\lambda^2 hk}}(6\lambda^2 hk + 3)}$$

$$= \lim_{\lambda\to 0} \frac{2\lambda hk + 4\lambda^3 h^2 k^2}{4\lambda^4 h^2 k^2 + 12\lambda^2 hk + 3} = \underline{0} \qquad \forall (h,k) \in \mathbb{R}^2$$

also: f in $(0,0)$ Gâteaux-diffbar.

Fréchet-Diffbarkeit in $(0,0)$:

vorab 2 Beobachtungen:

1) $e^x \geq 1+x \quad \forall x \in \mathbb{R}$

$\cdot)$ $x \leq -1$: trivial, da $e^x > 0 \quad \forall x \in \mathbb{R}$

$\cdot)$ $x > -1$: $e^x = \lim\limits_{n\to\infty} \left(1+\frac{x}{n}\right)^n \overset{\text{Bernoulli}}{\geq} 1+x$

2) $e^x \leq \frac{1}{1-x} \quad \forall |x|<1$

$$1+u = \frac{1-u}{1-u} + u = \frac{1-u^2}{1-u} \leq \frac{1}{1-u} \quad \forall |u|<1$$

$$\Rightarrow e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \leq \lim_{n\to\infty} \frac{1}{\left(1+\frac{x}{n}\right)^n} \overset{\text{Bernoulli}}{\leq} \frac{1}{1-x} \quad \forall |x|<1$$

ZZ: $\lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k)-f(0,0)-0|}{\|(h,k)\|} = 0$

$$\forall |hk|<1: \frac{\left|\frac{hk}{e^{hk}-1} - 1\right|}{\|(h,k)\|} = \frac{|hk-(e^{hk}-1)|}{|e^{hk}-1|\cdot\|(h,k)\|}$$

$$\overset{1)}{\leq} \frac{e^{hk}-1-hk}{|hk|\cdot\|(h,k)\|}$$

$$\overset{2)}{\leq} \frac{\frac{1}{1-hk}-1-hk}{|hk|\cdot\|(h,k)\|}$$

$$= \frac{h^2k^2}{|hk|\,(1-hk)\,\|(h,k)\|} \leq \frac{\|(h,k)\|}{1-hk} \to 0 \text{ für } (h,k)\to(0,0)$$

$\Rightarrow f$ in $(0,0)$ diffbar.

255) Funktionalmatrix:

$$\frac{\partial f}{\partial x}(x,y) = \frac{y(e^{xy}-1)-xy^2e^{xy}}{(e^{xy}-1)^2} = \frac{y}{e^{xy}-1} - \frac{xy^2e^{xy}}{(e^{xy}-1)^2} \qquad \forall xy \neq 0$$

$$\frac{\partial f}{\partial x}(x,0) = \lim_{\lambda\to 0}\frac{f(x+\lambda,0)-f(x,0)}{\lambda} = 0$$

$$\frac{\partial f}{\partial x}(0,y) = \lim_{\lambda\to 0}\frac{f(\lambda,y)-f(0,y)}{\lambda}$$

$$= \lim_{\lambda\to 0}\frac{\lambda y - e^{\lambda y}+1}{\lambda(e^{\lambda y}-1)} \quad „\frac{0}{0}“$$

$$= \lim_{\lambda\to 0}\frac{y-ye^{\lambda y}}{e^{\lambda y}-1+\lambda y e^{\lambda y}} \quad „\frac{0}{0}“$$

$$= \lim_{\lambda\to 0}\frac{-y^2\cancel{e^{\lambda y}}}{2y\cancel{e^{\lambda y}}+\lambda y^2\cancel{e^{\lambda y}}}$$

$$= \lim_{\lambda\to 0} -\frac{y}{2+\lambda} = -\frac{y}{2}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{\lambda \to 0} \frac{f(\lambda,0) - f(0,0)}{\lambda} = 0$$

aufgrund der Symmetrie folgt direkt:

$$\frac{\partial f}{\partial y}(x,y) = \frac{x}{e^{xy}-1} - \frac{x^2 y e^{xy}}{(e^{xy}-1)^2} \qquad \forall\, xy \neq 0$$

$$\frac{\partial f}{\partial y}(x,0) = -\frac{x}{2}$$

$$\frac{\partial f}{\partial y}(0,y) = \frac{\partial f}{\partial y}(0,0) = 0$$

$$\Rightarrow \frac{df}{d(x,y)}(x,y) = \left(\frac{y}{e^{xy}-1} - \frac{xy^2e^{xy}}{(e^{xy}-1)^2} \,,\, \frac{x}{e^{xy}-1} - \frac{x^2ye^{xy}}{(e^{xy}-1)^2} \right) \qquad xy \neq 0$$

$$\frac{df}{d(x,y)}(x,0) = \left(0 \,,\, -\frac{x}{2} \right)$$

$$\frac{df}{d(x,y)}(0,y) = \left(-\frac{y}{2} \,,\, 0 \right)$$

$$\frac{df}{d(x,y)}(0,0) = (0 \,,\, 0)$$

<u>Stetigkeit</u> an $(0,0)$:

$$\lim_{x \to 0} \frac{df}{d(x,y)}(x,0) = \lim_{y \to 0} \frac{df}{d(x,y)}(0,y) = (0,0)$$

$$\text{ZZ: } \lim_{(x,y)\to(0,0)} \frac{df}{d(x,y)}(x,y) = (0,0) \overset{\text{Symm.}}{\Leftrightarrow} \lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y) = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y) = \lim_{(x,y)\to(0,0)} y \left(\frac{1}{e^{xy}-1} - \frac{xye^{xy}}{(e^{xy}-1)^2} \right)$$

$$\overset{z:=xy}{=} \lim_{y \to 0} y \cdot \lim_{z \to 0} \left(\frac{e^z - 1 - ze^z}{(e^z-1)^2} \right)$$

$$\overset{\text{„}\frac{0}{0}\text{"}}{=} 0 \cdot \lim_{z \to 0} \frac{\cancel{e^z} - \cancel{e^z} - ze^{\cancel{z}}}{2(e^z-1)\cancel{e^z}}$$

$$\overset{\text{„}\frac{0}{0}\text{"}}{=} 0 \cdot \lim_{z \to 0} -\frac{1}{2e^z} = 0 \cdot -\frac{1}{2} = 0$$

also $\frac{df}{d(x,y)}$ an $(0,0)$ stetig, notwendigerweise in Umg. von $(0,0)$ beschränkt

$\Rightarrow f$ an $(0,0)$ stetig und diffbar.

248) $$f(x,y) = \begin{cases} \left(xy \cdot \sin\frac{1}{x^2+y^2},\ (x^2+y^2)\sin\frac{1}{xy}\right) & xy \neq 0 \\ (0,0) & xy = 0 \end{cases}$$

<u>1. Komponente</u>

Auf $\mathbb{R}^2 \setminus \{(0,0)\}$ ist f_1 als Zusammensetzung diffbarer Fkt. diffbar
$\Rightarrow$ alle Richtungsabl. existieren.

$$f'_{1(h,k)}(0,0) = \lim_{\lambda\to 0} \frac{f_1(\lambda h, \lambda k) - f(0,0)}{\lambda} = \lim_{\lambda \to 0} \frac{\lambda^{\cancel{2}} hk \cdot \sin\frac{1}{\lambda^2(h^2+k^2)}}{\cancel{\lambda}} = 0 \quad \forall (h,k) \in \mathbb{R}^2\setminus\{0\}$$

$$\frac{|f_1(h,k) - f_1(0,0) - 0|}{\|(h,k)\|} = \frac{|hk \cdot \sin\frac{1}{h^2+k^2}|}{\|(h,k)\|} \leq \frac{\|(h,k)\|^2_{max}}{\|(h,k)\|_{max}} = \|(h,k)\|_{max} \to 0 \quad \text{für } (h,k)\to(0,0)$$

<u>2. Komponente</u>

selbes Argument wie oben: f_2 diffbar $\forall (x,y) \in \mathbb{R}^2$ mit $xy \neq 0$

$$f'_{2(h,k)}(0,y) = \lim_{\lambda\to 0} \frac{f_2(0+\lambda h, y+\lambda k) - f_2(0,y)}{\lambda} = \begin{cases} 0 & h=0 \\ ? & h\neq 0 \end{cases}$$

$$h \neq 0 \Rightarrow f'_{2(h,k)}(0,y) = \lim_{\lambda\to 0} \frac{(\lambda^2 h^2 + (y+\lambda k)^2)\sin\frac{1}{\lambda h(y+\lambda k)}}{\lambda}$$

$$h \neq 0 \quad \Rightarrow f'_{2(h,k)}(0,y)\ \exists \Leftrightarrow \lim_{\lambda \to 0} \frac{\lambda^2 h^2 + y^2 + 2y\lambda k + \lambda^2 k^2}{\lambda} = 0 \Leftrightarrow y = 0$$

~~analog: $f'_{2(h,k)}(x,0)\ \exists \Leftrightarrow x=0$~~

$$\Rightarrow f'_{2(h,k)}(0,y) = \begin{cases} 0 & h=0 \vee y=0 \\ \nexists & \text{sonst} \end{cases} \qquad \text{analog: } f'_{2(h,k)}(x,0) = \begin{cases} 0 & k=0 \vee x=0 \\ \nexists & \text{sonst} \end{cases}$$

$\Rightarrow$ für f_2 $\exists$ in $(0,0)$ alle Richtungsableitungen, $f'_{2(h,k)}(0,0) = 0 \quad \forall (h,k) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$\frac{|f_2(h,k) - f_2(0,0) - 0|}{\|(h,k)\|} = \frac{|(h^2+k^2)\sin\frac{1}{hk}|}{\|(h,k)\|} \leq \frac{2\|(h,k)\|^2_{max}}{\|(h,k)\|_{max}} \to 0 \quad \text{für } (h,k) \to (0,0)$$

also f_2 in $(0,0)$ auch Fréchet-diffbar.

Insgesamt: f diffbar $\forall (x,y) \in \{(\mathbb{R}\times\mathbb{R}) \mid xy \neq 0\} \cup \{(0,0)\}$

260) <u>Funktionalmatrix</u>

<u>1. Komponente:</u>

$$\frac{\partial f_1}{\partial x}(x,y) = y \cdot \sin\frac{1}{x^2+y^2} + xy \cdot \cos\frac{1}{x^2+y^2} \cdot \frac{2x}{(x^2+y^2)^2}$$
$$= y \cdot \sin\frac{1}{x^2+y^2} + \frac{2x^2 y}{(x^2+y^2)^2} \cdot \cos\frac{1}{x^2+y^2} \qquad (x,y) \neq (0,0)$$

$$\frac{\partial f_1}{\partial x}(0,0) = \lim_{\lambda \to 0} \frac{f(\lambda,0) - f(0,0)}{\lambda} = 0$$

analog wegen der Symmetrie:

$$\frac{\partial f_1}{\partial y}(x,y) = x \cdot \sin\frac{1}{x^2+y^2} + \frac{2xy^2}{(x^2+y^2)^2} \cdot \cos\frac{1}{x^2+y^2} \qquad (x,y) \neq (0,0)$$

$$\frac{\partial f_1}{\partial y}(0,0) = 0$$

2. Komponente:

$$\frac{\partial f_2}{\partial x}(x,y) = 2x \sin\frac{1}{xy} + (x^2+y^2)\cos\frac{1}{xy} \cdot -\frac{y}{x^2y^2}$$

$$= 2x \sin\frac{1}{xy} - \left(\frac{1}{y} + \frac{y}{x^2}\right)\cos\frac{1}{xy} \qquad xy \neq 0$$

$$\frac{\partial f_2}{\partial x}(x,0) = f'_{(1,0)}(x,0) \overset{24)}{=} 0 \qquad x \neq 0$$

$$\frac{\partial f_2}{\partial x}(0,y) = f'_{(1,0)}(0,y) \overset{24)}{=} \nexists \qquad y \neq 0$$

$$\frac{\partial f_2}{\partial x}(0,0) \overset{24)}{=} 0$$

analog wegen der Symmetrie:

$$\frac{\partial f_2}{\partial y}(x,y) = 2y \sin\frac{1}{xy} - \left(\frac{1}{x} + \frac{x}{y^2}\right)\cos\frac{1}{xy} \qquad xy \neq 0$$

$$\frac{\partial f_2}{\partial y}(x,0) = f'_{(0,1)}(x,0) = \nexists \qquad x \neq 0$$

$$\frac{\partial f_2}{\partial y}(0,y) = \frac{\partial f_2}{\partial y}(0,0) = 0 \qquad \cancel{y \neq 0}$$

$$\Rightarrow \frac{df}{d(x,y)}(x,y) = \begin{pmatrix} y \sin\frac{1}{x^2+y^2} + \frac{2x^2y}{(x^2+y^2)^2}\cos\frac{1}{x^2+y^2}, & x\sin\frac{1}{x^2+y^2} + \frac{2xy^2}{(x^2+y^2)^2}\cos\frac{1}{x^2+y^2} \\ 2x\sin\frac{1}{xy} - \left(\frac{1}{y}+\frac{y}{x^2}\right)\cos\frac{1}{xy}, & 2y\sin\frac{1}{xy} - \left(\frac{1}{x}+\frac{x}{y^2}\right)\cos\frac{1}{xy} \end{pmatrix} \quad xy \neq 0$$

$$\frac{df}{d(x,y)}(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Beschränktheit, Stetigkeit in $(0,0)$:

wähle Annäherung $x_n = y_n = \frac{1}{2\sqrt{\pi n}} \to 0$ für $n \to \infty$

$$\Rightarrow \lim_{n\to\infty} \frac{\partial f_1}{\partial x}(y_n,y_n) = \lim_{n\to\infty} y_n \underbrace{\sin\frac{1}{2y_n^2}}_{=0} + \frac{2y_n^3}{(2y_n^2)^2}\underbrace{\cos\frac{1}{2y_n^2}}_{=1}$$

$$= \lim_{n\to\infty} \frac{1}{2y_n} = \infty$$

also folgt 1) $\frac{\partial f_1}{\partial x}(x,y)$ ist in keiner Umgebung von $(0,0)$ beschränkt, die hinreichend nahe an $(0,0)$ beliebig große Funktionswerte aufweisen.

2) $\frac{\partial f_1}{\partial x}(x,y)$ ist in $(0,0)$ unstetig.

analog für $\frac{\partial f_1}{\partial y}(x,y)$ wegen der Symmetrie.

$$\lim_{n\to\infty} \frac{\partial f_2}{\partial x}(x_n,x_n) = \lim_{n\to\infty} 2x_n \underbrace{\sin\frac{1}{x_n^2}}_{=0} - \underbrace{\cos\frac{1}{x_n^2}}_{=1} \underbrace{\left(\frac{1}{x_n} + \frac{x_n}{x_n^2}\right)}_{=\frac{2}{x_n}} = \lim_{n\to\infty} -\frac{2}{x_n} = \infty$$

also auch $\frac{\partial f_2}{\partial x}(x,y)$ bzw. $\frac{\partial f_2}{\partial y}(x,y)$ in $(0,0)$ weder stetig noch beschränkt.

(obwohl f in $(0,0)$ stetig und diffbar!)