Analysis UE

III, 20, 24, 32, 36, 40, 48, 52

20)
$$\int_{-1}^{1} (x^5+x^2)\, e^{x^3}\, dx = \left| \begin{array}{l} x^3 = u \\ 3x^2\, dx = du \end{array} \right|$$
$$= \int_{\sqrt[3]{-1}}^{\sqrt[3]{1}} \frac{x^5+x^2}{3x^2}\, e^u\, du$$
$$= \int_{-1}^{1} \left(\frac{1}{3}x^3 + \frac{1}{3}\right) e^u\, du$$
$$= \frac{1}{3}\int_{-1}^{1} \underbrace{(u+1)}_{g(u)}\, \underbrace{e^u}_{f'(u)}\, du$$
$$= \frac{1}{3}\left((u+1)e^u \Big|_{-1}^{1} - \int_{-1}^{1} e^u\, du \right)$$
$$= \frac{1}{3}\left(2e - e^u\Big|_{-1}^{1}\right) = \frac{1}{3}(2e - e + e^{-1}) = \frac{e}{3} + \frac{e^{-1}}{3}$$

24) Sei $[p] = k$. Notation: $p_k(x) :=$ Polynom mit Grad k
$$\int p(x) \cos ax\, dx = \left| \begin{array}{l} f'(x) = \cos ax \\ g(x) = p(x) \end{array} \right|$$
$$= \underbrace{\frac{1}{a} p(x)}_{q_k(x)} \sin ax - \frac{1}{a}\int p'(x) \sin ax\, dx + c$$
$$= \left| \begin{array}{l} f'(x) = \sin ax \\ g(x) = p'(x) \end{array} \right|$$
$$= q_k(x) \sin ax + \underbrace{\frac{1}{a^2} p'(x)}_{r_{k-1}(x)} \cos ax + \frac{1}{a^2}\int p''(x) \cos ax\, dx + c$$

Nach k-maliger Ausführung kann direkt integriert werden und es folgt die Darstellung:
$$\int p(x) \cos ax\, dx = \left(q_k(x) + q_{k-2}(x) + \dots\right) \sin ax + \left(r_{k-1}(x) + r_{k-3}(x) + \dots\right) \cos ax + c$$
$$= q(x) \sin ax + r(x) \cos ax + c$$
wobei $[q] = [p]$ und $[r] = [p] - 1$

32) $\int \frac{x^2}{1-x^4}\,dx$

Partialbruchzerlegung:

$$\frac{x^2}{1-x^4} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{Cx+D}{1+x^2}$$

$$x^2 = A(1-x)(1+x^2) + B(1+x)(1+x^2) + (Cx+D)(1-x^2)$$

x^0: $0 = A+B+D$
x^1: $0 = -A+B+C$
x^2: $1 = A+B-D$
x^3: $0 = -A+B-C$

$\Rightarrow D = -\frac{1}{2},\ C=0,\ A=B=\frac{1}{4}$

$$\int \frac{x^2}{1-x^4}\,dx = \frac{1}{4}\int \frac{1}{1+x}\,dx + \frac{1}{4}\int \frac{1}{1-x}\,dx - \frac{1}{2}\int \frac{1}{1+x^2}\,dx$$

$$= \frac{\ln|1+x|}{4} + \frac{\ln|1-x|}{4} - \frac{\arctan x}{2} + C$$

36) $\int \frac{x^6-1}{x^6+x^3}\,dx = \int \frac{\cancel{(x^3+1)}(x^3-1)}{x^3\cancel{(x^3+1)}}\,dx$

$$= \int 1\,dx - \int \frac{1}{x^3}\,dx$$

$$= x + \frac{1}{2x^2} + C$$

40) $$\int \frac{dx}{x^5(x^8+1)^2} = \left| \begin{matrix} x^4 = u \\ 4x^3\,dx = du \end{matrix} \right|$$

$$= \int \frac{du}{x^5(u^2+1)^2\; 4x^3}$$

$$= \int \frac{du}{4u^2\,(u^2+1)^2}$$

$$= \left| \begin{matrix} \frac{1}{4u^2} = f'(u) \\ \frac{1}{(u^2+1)^2} = g(u) \end{matrix} \right|$$

$$= -\frac{1}{4u}\cdot\frac{1}{(u^2+1)^2} - \int -\frac{\cancel{1}}{\cancel{4}u}\cdot\frac{\cancel{-2}\cancel{(u^2+1)}\cancel{2u}}{(u^2+1)^{\cancel{4}\,3}} + c$$

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$$= \text{———}\,''\,\text{———} - \left(\frac{u}{4(u^2+1)^2} + \frac{3}{4}\left(\frac{u}{2(u^2+1)} + \frac{1}{2}\int\frac{du}{u^2+1}\right)\right) + c$$

$$= \text{———}\,''\,\text{———} - \frac{u}{4(u^2+1)^2} - \frac{3u}{8(u^2+1)} - \frac{3}{8}\arctan u + c$$

$$= \frac{(-2) - 2u^2 - 3u^2(u^2+1)}{8u\,(u^2+1)^2} - \frac{3}{8}\arctan u + c$$

$$= -\frac{2\cancel{(1+u^2)} + 3u^2\cancel{(u^2+1)}}{8u\,(u^2+1)^{\cancel{2}}} - \frac{3}{8}\arctan u + c$$

$$= -\frac{2+3x^8}{8x^4(x^8+1)} - \frac{3}{8}\arctan x^4 + c$$

48) $$\int \frac{dx}{x^7\sqrt{x^6+1}} = \left| \begin{matrix} x^3 = u \\ 3x^2\,dx = du \end{matrix} \right|$$

$$= \frac{1}{3}\int \frac{du}{u^3\sqrt{u^2+1}}$$

$$= \left| \begin{matrix} t = \frac{1}{u} \\ dt = -\frac{1}{u^2}du = -t^2\,du \end{matrix} \right|$$

$$= \frac{1}{3}\int \frac{t^{\cancel{3}}}{\sqrt{(\frac{1}{t})^2+1}}\cdot\frac{dt}{\cancel{-t^2}\,-1}$$

$$= \frac{1}{3}\int \frac{-t}{\sqrt{\frac{1+t^2}{t^2}}}\,dt$$

$$= (\operatorname{sgn} t)\cdot\frac{1}{3}\int\frac{-t^2}{\sqrt{1+t^2}}\,dt$$

Auflösung mit dem 2. Hermite-Ansatz:

$$\int\frac{-t^2}{\sqrt{t^2+1}}\,dt = (Q_1 t + Q_0)\sqrt{t^2+1} + R\int\frac{dt}{\sqrt{t^2+1}} + c$$

Differenzieren:

$$\frac{-t^2}{\sqrt{t^2+1}} = Q_1\sqrt{t^2+1} + (Q_1 t + Q_0)\frac{1}{\cancel{2}}(t^2+1)^{-\frac{1}{2}}\cancel{2}t + \frac{R}{\sqrt{t^2+1}}$$

Koeffizientenvergleich:

$$-t^2 = Q_1(2t^2+1) + Q_0 t + R$$

$$\begin{aligned} t^2&: \ -1 = 2Q_1 \quad \Rightarrow Q_1 = -\tfrac{1}{2} \\ t&: \ 0 = Q_0 \\ t^0&: \ 0 = Q_1 + R \quad \Rightarrow R = \tfrac{1}{2} \end{aligned}$$

$$\Rightarrow \frac{(\operatorname{sgn} t)}{3} \int \frac{-t^2}{\sqrt{t^2+1}}\, dt = \frac{(\operatorname{sgn} t)}{\cancel{3}} \left(-\frac{\sqrt{t^2+1}\, t}{6} + \frac{1}{6} \overbrace{\int \frac{dt}{\sqrt{t^2+1}}}^{= \operatorname{arsinh} t + c} \right) + c$$

$$= \left(\operatorname{sgn} \tfrac{1}{u}\right) \left(-\frac{\sqrt{\frac{1}{u^2}+1}}{6u} + \frac{1}{6} \operatorname{arsinh} \frac{1}{u} \right) + c$$

$$= (\operatorname{sgn} u) \left(-\frac{\sqrt{\frac{1+u^2}{u^2}}}{6u} + \text{——"——} \right) + c$$

$$= -\frac{\sqrt{1+u^2}}{6u^2} + (\operatorname{sgn} u)\ \text{——"——} + c$$

$$= -\frac{\sqrt{1+x^6}}{6x^6} + (\operatorname{sgn} x) \frac{1}{6} \operatorname{arsinh} \frac{1}{x^3} + c$$

52)
$$\int (x^2+3x-3)\sqrt{-x^2-3x+7}\, dx \overset{\text{2. Hermite-A.}}{=} (Q_3x^3+Q_2x^2+Q_1x+Q_0)\sqrt{-x^2-3x+7} + R\int \frac{dx}{\sqrt{-x^2-3x+7}} + c$$

Diff.: $$(x^2+3x-3)\sqrt{-x^2-3x+7} = (3Q_3x^2+2Q_2x+Q_1)\sqrt{-x^2-3x+7} + (Q_3x^3+Q_2x^2+Q_1x+Q_0)\cdot \frac{1}{2}(-x^2-3x+7)^{-\frac{1}{2}}(-2x-3) + \frac{R}{\sqrt{-x^2-3x+7}}$$

$$\Leftrightarrow (x^2+3x-3)(-x^2-3x+7) = (3Q_3x^2+2Q_2x+Q_1)(-x^2-3x+7) + (Q_3x^3+Q_2x^2+Q_1x+Q_0)\left(-x-\tfrac{3}{2}\right) + R$$

Koeff.-Vgl.:

$$\begin{aligned} x^4&: \ -1 = -3Q_3 - Q_3 \\ x^3&: \ -6 = -9Q_3 - 2Q_2 - \tfrac{3}{2}Q_3 - Q_2 = -\tfrac{21}{2}Q_3 - 3Q_2 \\ x^2&: \ 1 = -Q_1 - 6Q_2 + 21Q_3 - Q_1 - \tfrac{3}{2}Q_2 = 21Q_3 - \tfrac{15}{2}Q_2 - 2Q_1 \\ x&: \ 30 = -3Q_1 + 14Q_2 - \tfrac{3}{2}Q_1 - Q_0 = 14Q_2 - \tfrac{9}{2}Q_1 - Q_0 \\ x^0&: \ -21 = 7Q_1 - \tfrac{3}{2}Q_0 + R \end{aligned}$$

$$\Rightarrow Q_3 = \frac{1}{4}, \quad Q_2 = \frac{9}{8}, \quad Q_1 = -\frac{67}{32}, \quad Q_0 = -\frac{309}{64}, \quad R = -\frac{1739}{128}$$

$$\Rightarrow \int (x^2+3x-3)\sqrt{-x^2-3x+7}\, dx = \left(\frac{1}{4}x^3 + \frac{9}{8}x^2 - \frac{67}{32}x - \frac{309}{64}\right)\sqrt{-x^2-3x+7} - \frac{1739}{128} \underbrace{\int \frac{dx}{\sqrt{x^2-3x+7}}}_{\text{Aufl. nach S.37 oben}} + c$$

$$\int \frac{dx}{\sqrt{-x^2-3x+7}} = \begin{vmatrix} y = x + \frac{3}{2} \\ dy = dx \end{vmatrix}$$

$$= \int \frac{dy}{\sqrt{\frac{37}{4} - y^2}}$$

$$= \int \frac{dy}{\sqrt{\frac{37}{4}}\sqrt{1 - \frac{4}{37}y^2}}$$

$$= \begin{vmatrix} z^2 = \frac{4}{37}y^2 \\ dz = \sqrt{\frac{4}{37}}\,dy \end{vmatrix}$$

$$= \int \frac{dz}{\cancel{\sqrt{\frac{37}{4}}}\sqrt{1-z^2}\cancel{\sqrt{\frac{4}{37}}}}$$

$$= \arcsin z + C = \arcsin\left(\frac{2}{\sqrt{37}}y\right) + C = \arcsin\left(\frac{\cancel{2}}{\sqrt{37}} \cdot \frac{2x+3}{\cancel{2}}\right) + C$$

$$\Rightarrow \int (x^2+3x-3)\sqrt{-x^2-3x+7}\,dx = \left(\frac{1}{4}x^3 + \frac{9}{8}x^2 - \frac{67}{32}x - \frac{309}{64}\right)\sqrt{-x^2-3x+7}$$

$$- \frac{1739}{128}\arcsin\left(\frac{2x+3}{\sqrt{37}}\right) + C$$